A PREFACE TO COSMOGONY. I. THE ENERGY EQUATION AND THE VIRIAL THEOREM FOR COSMIC DISTRIBUTIONS

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ABSTRACT

This paper deals with the dynamics of an unbounded, statistically homogeneous, and isotropic distribution of gravitating particles. As is well known, the mean density and velocity fields depend on a single function, the cosmic scale factor $R(t)$, given by relativistic cosmology. In an approximation that is adequate for all astronomical applications, the peculiar velocity $\boldsymbol{v}$ of a test particle is shown to satisfy the equations of motion

$$\frac{d}{dt}(R\boldsymbol{v}) = -R\frac{\partial\varphi}{\partial\boldsymbol{x}},$$

where the potential φ is related to the peculiar density field σ by Poisson's equation. In a uniform medium ($\sigma \equiv 0$) the velocity of a test particle decays as R^{-1}. More generally, the form of the equations of motion implies that "initial" conditions cannot have an enduring influence on particle motions, which must accordingly be caused mainly by the action of the fluctuating gravitational force field $\partial\varphi/\partial\boldsymbol{x}$.

This idea finds its mathematical expression in two coupled equations for $T_m = \frac{1}{2}\langle v^2\rangle_m$, the mean peculiar kinetic energy per unit mass, and $U_m = \frac{1}{2}\langle\varphi\rangle_m$, the mean peculiar potential energy per unit mass:

$$\frac{d}{dt}(T_m + U_m) + \frac{\dot{R}}{R}(2T_m + U_m) = 0,$$

$$2T_m(t) + U_m(t-\epsilon) = 0,$$

where $\epsilon \ll t$. The first equation (the cosmologic energy equation) was first obtained in this form by Irvine. It reduces to a more familiar form if one sets

$$\bar{e} = \bar{\rho}(T_m + U_m),$$

$$3\bar{p} = \bar{\rho}(2T_m + U_m),$$

where $\bar{e}$ is the mean internal energy density and $\bar{p}$ the cosmic pressure. The second equation (the cosmologic virial theorem) is here derived by a fundamentally different method from the one used to derive the virial theorem for a bounded system. The present method applies also to bounded systems, but the conventional method breaks down when applied to an unbounded system.

The cosmologic virial theorem is expected to be valid as long as non-gravitational forces and the effects of radiation are negligible. The specific energies T_m and U_m are then nearly constant in time The constancy of U_m, which may be written in the form

$$U_m = -\pi G a^2 \bar{\rho}\lambda^2,$$

where λ is a clustering scale and $a^2 = \langle\sigma^2\rangle/\bar{\rho}^2$ is a measure of the density contrast, implies that a^2 increases at least as fast as R—a conclusion reached previously (Layzer 1954*a*) by a less rigorous argument.

I. INTRODUCTION

This is the first in a series of papers expounding an approach to cosmogony whose central idea is that self-gravitating systems are formed through gravitational clustering in an expanding cosmic distribution, rather than through the fragmentation of finite gas clouds.[1] The hypothesis of gravitational clustering depends for its theoretical justification chiefly on the results obtained in the present paper. These results are, on the one hand,

[1] The fragmentation hypothesis has been criticized in a previous paper (Layzer 1963).

direct consequences of Einstein's theory of gravitation as applied to a statistically homogeneous and isotropic distribution of mass points. On the other hand, they admit of empirical verification through observations that are probably within the scope of current techniques. The present paper is therefore less speculative than its successors, now in preparation, which will deal with the origin of the clustering hierarchy, the structure of galaxies, and the origin of planetary systems.

It is a plausible conjecture that the peculiar motions of galaxies are caused by local variations in the gravitational field due to clustering. If we could assume that all galaxies belong to compact clusters and if we had a sufficiently complete statistical description of the clusters, the virial theorem would enable us to express this conjecture in quantitative form. In fact, only a small proportion of galaxies belong to recognizable clusters, and, although many clusters undoubtedly go unrecognized, the detailed consequences of the assumption that *all* galaxies belong to clusters, as worked out by Neyman and Miss Scott (1952), appear to conflict with observation (see Neyman and Scott 1956, p. 91). If we adopt a less restrictive model for the spatial distribution of galaxies (see, e.g., Layzer 1956*a*), we need to have a version of the virial theorem that applies to a cosmic distribution of interacting particles, i.e., an unbounded, expanding distribution characterized by statistical homogeneity and isotropy.

Irvine (1961) has shown that the standard derivation of the virial theorem, when applied to a finite sample of a cosmic distribution, leads to a trivial identity. In retrospect this result is not surprising. The classical derivation of the virial theorem rests on the assumption that the system under consideration occupies, on the average, a fixed volume of space. An isolated system can fulfil this condition if and only if the forces that hold it together are related in a particular way to its internal motions. In a cosmic distribution, however, any finite subsystem whose dimensions are large compared with the scale of clustering occupies an ever increasing volume, whose expansion is uninfluenced, except through surface effects, by local gravitational fields and peculiar motions. This does not imply that in a cosmic distribution the local gravitational fields and the peculiar motions are unrelated. Rather, it suggests that the conventional mathematical expression of the relationship is inappropriate. This is also suggested by the following more general consideration: In describing a bounded physical system, one normally adopts at the outset a definite origin of co-ordinates and identifies it with the center of mass of the system. Since a cosmic distribution has neither a center of mass nor any other preferred point, the choice of a definite origin introduces an artificial asymmetry into its description.

In order to avoid specifying a definite origin, one must work with displacements rather than with position vectors and with velocities relative to an instantaneous local standard of rest rather than with velocities in a fixed frame of reference. In addition, one must use a description of the local gravitational fields that is invariant under translation. The gravitational field equations automatically satisfy this requirement but the conventional auxiliary conditions do not, since they refer to the behavior of the field at great distances from some fixed point. One can resolve this difficulty by using statistical symmetry conditions instead of boundary conditions. In the quasi-Newtonian approximation on which the following work is based, such symmetry conditions ensure that the fluctuating component of the gravitational field is uniquely determined by the distribution of matter.[2]

The restrictions just outlined do not in themselves provide a solution of the problem in hand, but they define the framework within which the solution must be sought. In the next section we shall derive the fundamental equations governing local irregularities in a cosmic distribution, verifying results previously obtained by Irvine (1961) and Layzer (1954*a*). This will prepare the way for a derivation of the virial theorem in Section

[2] By contrast, the mean gravitational field is not completely determined by the mean distribution of matter and symmetry conditions; the mean spatial curvature at any given instant can still be freely specified. This suggests that relativistic cosmology in its present form is incomplete and needs to be supplemented in some as yet unknown manner.

III. At the end of Section III we shall discuss briefly the possibility of testing the cosmologic virial theorem observationally. The data needed for such a test appear to lie within the scope of current observational techniques.

II. THE EQUATIONS OF MOTION AND THE ENERGY EQUATION

Let $\rho(x, t)$ denote the density of matter in a statistically homogeneous and isotropic distribution. We may regard ρ either as a random function of position or as a particular realization of a random function. If we adopt the first point of view, the statistical properties of the density field will be described in terms of what the physicist calls ensemble averages; if we adopt the second point of view, in terms of space averages. If the two averages coincide, the distribution is said to be ergodic; we shall assume that this is the case.[3] In what follows, the notation $\langle \quad \rangle$ may be interpreted as indicating either an ensemble average or a space average. In either case, we assume that the averaging process is a linear operation.

The density function may be written in the form

$$\rho(x, t) = \bar{\rho}(t) + \sigma(x, t) , \tag{1}$$

where

$$\langle\rho(x, t)\rangle = \bar{\rho}(t) , \qquad \langle\sigma(x, t)\rangle = 0 . \tag{2}$$

The autocorrelation function f of the fluctuation field σ is defined by

$$\langle\sigma(x, t)\sigma(x', t)\rangle = \langle\sigma^2\rangle f(|x - x'|) . \tag{3}$$

With the help of the autocorrelation function, one can define various clustering scales, the most important of which in the present context is given by

$$\int_0^\infty f(r)\, r\, dr = \lambda^2 . \tag{4}$$

We assume that λ is finite.

In a first approximation the geometry of space-time is that appropriate to a uniform fluid (called the "substratum") whose density and pressure coincide at all times with the mean density and mean pressure of the actual cosmic distribution. The metric of the substratum is given by

$$d\tau^2 = dt^2 - R^2(t)A^2(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) , \qquad A = (1 + \tfrac{1}{4}k\bar{r}^2)^{-1} , \tag{5}$$

where t is the cosmic time and the barred quantities are co-moving co-ordinates; the unbarred space co-ordinates of equations (1)–(3) are related to the co-moving co-ordinates by

$$x = R(t)\bar{x} . \tag{6}$$

The co-ordinate system (x, t) is locally inertial; in it the laws of local physics take their customary forms. The scale of the co-moving co-ordinates can always be chosen so as to make the curvature parameter k have one of the three values $+1$, 0, -1. We shall assume that if $k \neq 0$, then $\lambda/R(t) \ll 1$, so that Euclidean geometry prevails to a good approximation in a region whose diameter is a few times the scale of local irregularities.

The cosmic scale factor R and the mean density $\bar{\rho}$ are connected by the relation

$$R^3\bar{\rho} = \text{Const.} , \tag{7}$$

[3] Ergodic theory deals nominally with stationary stochastic processes. However, statements about stationary stochastic processes can readily be translated into statements about statistically homogeneous and isotropic spatial distributions.

which is the equation of continuity for the substratum.[4] Let v denote the velocity of a particle relative to the substratum; clearly,

$$\langle v \rangle = 0 . \tag{8}$$

The equation of continuity for the actual distribution is[5]

$$\frac{\partial}{\partial t}(R^3 \rho) + \frac{\partial}{\partial x} \cdot (R^3 \rho v) = 0 , \tag{9}$$

which, by virtue of equation (7), reduces to

$$\frac{\partial}{\partial t}(R^3 \sigma) + \frac{\partial}{\partial x} \cdot (R^3 \rho v) = 0 . \tag{10}$$

Equation (9) equates the rate of change of the mass enclosed by a surface every point of which is locally at rest (i.e., at rest with respect to the substratum) to the flux of mass into the expanding region bounded by this surface.

The peculiar potential φ is defined as that solution of Poisson's equation,

$$\nabla^2 \varphi = 4\pi G \sigma , \tag{11}$$

which satisfies the conditions of statistical homogeneity and isotropy and the convention

$$\langle \varphi \rangle = 0 . \tag{12}$$

These conditions determine φ uniquely; it is given by

$$\varphi(x, t) = -G \int \frac{\sigma(x', t)\, dV'}{|x - x'|} . \tag{13}$$

The integration extends formally over all space, but in practice only the contributions from within a few multiples of the clustering scale λ are significant. In this region Euclidean geometry is valid and

$$dV' = dx'dy'dz' . \tag{14}$$

We are now in a position to write down the equations of motion for a test particle. Let V denote the velocity of a particle *relative to a fixed origin*, which is locally at rest. The acceleration of a particle that coincides instantaneously with the origin is given in first approximation by[6]

$$\frac{\partial V}{\partial t} = -\frac{\partial \varphi}{\partial x} . \tag{15}$$

[4] In eq. (7), as in all the subsequent work, all terms of order $(v/c)^2$ compared with the leading terms are to be neglected. In particular, it is not necessary in the present discussion to distinguish between the matter-energy density ρ and the rest-mass density μ.

[5] In co-moving co-ordinates the equation of continuity takes the form

$$\frac{\partial}{\partial t}(R^3 \rho) + \frac{\partial}{\partial \bar{x}}\left(R^3 \rho \frac{d\bar{x}}{dt}\right) = 0 .$$

On setting $v = R(d\bar{x}/dt)$ and noting that $\partial/\partial x = 1/R(t)\ \partial/\partial \bar{x}$, one obtains equation (9), *provided that $\partial/\partial t$ is interpreted as a time-derivative with $\bar{x}$ (not x) held constant* In this paper $\partial/\partial t$ always has this meaning In other words, x should always be interpreted as a function of $\bar{x}$ and t (as in the above expression for $\partial/\partial x$) rather than as an independent variable.

[6] For a detailed mathematical discussion of this approximation, together with a complete list of references, see Irvine (1961).

In accordance with the program outlined in Section I, we must replace equation (15) by a relation involving $\boldsymbol{v}$ instead of $\boldsymbol{V}$. To derive this relation, we note that a particle that is at the origin at time t will be at the point $d\boldsymbol{x} = \boldsymbol{v}dt$ at time $t + dt$. Here the velocity of the substratum, as seen from the origin, is $H(t + dt)d\boldsymbol{x} = H(t)\boldsymbol{v}(t)dt$, to first order, where

$$H(t) = \frac{\dot{R}(t)}{R(t)}. \tag{16}$$

Hence, in view of equation (15), the rate of change of the peculiar velocity will be given by

$$\frac{\partial \boldsymbol{v}}{\partial t} = -H\boldsymbol{v} - \frac{\partial \varphi}{\partial \boldsymbol{x}}, \tag{17}$$

or, equivalently,

$$\frac{\partial}{\partial t}(R\boldsymbol{v}) = -\frac{\partial}{\partial \boldsymbol{x}}(R\varphi). \tag{18}$$

As equations (17) and (18) make no reference to a particular origin, they are valid at every point. Equation (18) shows that in a uniform distribution ($\sigma \equiv \varphi \equiv 0$) the peculiar velocity of every particle varies as R^{-1}. Hence in such a distribution the cosmic pressure and the internal energy density, both of which are proportional to $\bar{\rho}\langle v^2\rangle$, vary as R^{-5}—a well-known result.

To derive the energy equation for local irregularities, we form the scalar product of equation (18) with $R\boldsymbol{v}\rho dV$ and integrate over the distribution, which in the first instance we assume to be finite ($k = 1$). We also assume that the distribution consists of discrete particles, whose internal structure we shall ignore. The left side of the resulting equation, then, is

$$\tfrac{1}{2}\int \frac{\partial}{\partial t}(Rv)^2 \rho dV = \tfrac{1}{2}\sum m \frac{\partial}{\partial t}(Rv)^2 = \frac{d}{dt}\left(R^2 \sum \tfrac{1}{2} m v^2\right) = \frac{d}{dt}(R^2 T), \tag{19}$$

where T denotes the total kinetic energy associated with peculiar motions. We transform the right side by partial integration and then use equations (10) and (13):

$$\begin{aligned} -\int R\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}(R\varphi)\rho dV &= R^2 \int \left[\frac{\partial}{\partial \boldsymbol{x}}\cdot(\rho \boldsymbol{v})\right]\varphi dV \\ &= R^2 G \int \frac{\partial}{\partial t}(\sigma dV)\int \frac{\sigma' dV'}{|\boldsymbol{x}-\boldsymbol{x}'|} = R^2 G \int \frac{\partial}{\partial t}(\sigma' dV')\int \frac{\sigma dV}{|\boldsymbol{x}-\boldsymbol{x}'|}. \end{aligned} \tag{20}$$

Here $\sigma \equiv \sigma(\boldsymbol{x}, t)$, $\sigma' \equiv \sigma(\boldsymbol{x}', t)$, etc. The potential energy associated with the fluctuating density field σ is given by

$$U = -\tfrac{1}{2}G\int\int \frac{\sigma(\boldsymbol{x}, t)\sigma(\boldsymbol{x}'t)dVdV'}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{21}$$

Since, by equations (6) and (16),

$$\frac{\partial}{\partial t}\frac{1}{|x - x'|} = -\frac{H}{|x - x'|}, \tag{22}$$

the right side of the energy equation finally takes the form

$$-R^2\left(\frac{d}{dt} + H\right)U. \tag{23}$$

Equating expressions (19) and (23), we obtain

$$\frac{d}{dt}(T+U)+H(2T+U)=0, \tag{24}$$

an equation first derived in this form by Irvine (1961).

The foregoing derivation does not immediately apply to an infinite distribution, but the necessary modifications are essentially trivial: We extend the integration over a finite volume V whose boundary is locally at rest, divide through by $\bar{\rho}V$, the mean mass of the region, and let V increase indefinitely. The "surface contributions" all vanish in the limit $V = \infty$. We shall use the suffix m to distinguish quantities defined per unit mass, and the notation $\langle\ \rangle_m$ to indicate a mass average. Thus

$$T_m=\lim_{V\uparrow}(\bar{\rho}V)^{-1}\int_V \tfrac{1}{2}\rho v^2 dV=\tfrac{1}{2}\langle v^2\rangle_m, \tag{25}$$

$$\begin{aligned} U_m &= \lim_{V\uparrow}(\bar{\rho}V)^{-1}\int_V \tfrac{1}{2}\sigma\varphi dV \\ &= \lim_{V\uparrow}(\bar{\rho}V)^{-1}\int_V \tfrac{1}{2}\rho\varphi dV=\tfrac{1}{2}\langle\varphi\rangle_m . \end{aligned} \tag{26}$$

In place of equation (24) we now have

$$\frac{d}{dt}(T_m+U_m)+H(2T_m+U_m)=0, \tag{27}$$

which is valid for all three possible values of the curvature parameter k.

The potential energy per unit mass U_m can be written in the useful form

$$U_m = -\pi G\alpha^2\bar{\rho}\lambda^2 , \tag{28}$$

where

$$\langle\sigma^2\rangle = \alpha^2\bar{\rho}^2 . \tag{29}$$

Equation (28)—with a slightly different definition of λ—has been given previously (Layzer 1954*a*).

On comparing equation (27) with the cosmologic energy equation

$$\frac{d(\bar{e}R^3)}{dt}+\bar{p}\frac{dR^3}{dt}=0, \tag{30}$$

in which $\bar{e}$ denotes the internal energy density and $\bar{p}$ the cosmic pressure, we see that the two equations will coincide (see eq. [16]) if we set

$$\bar{e} = \bar{\rho}(T_m + U_m) , \tag{31}$$

$$3\bar{p} = \bar{\rho}(2T_m + U_m) . \tag{32}$$

Irvine (1961) has given a more complete justification for these identifications. Although equations (31) and (32) have precisely the forms suggested by statistical theories of imperfect gases and liquids, the gravitational contributions, which arise from non-uniformity of the spatial distribution, have usually been neglected in cosmology.

Equation (30) is usually taken to mean that any volume V of the substratum expands adiabatically, doing work against the rest of the universe at the rate $\bar{p}(dV/dt)$. The present derivation leads one to interpret the term $\bar{p}(dV/dt)$ as describing a uniformly distributed source or sink of energy having a purely kinematic origin. That the energy associated with local irregularities is not in general conserved is a consequence of the non-Newtonian character of the equations of motion. This point is discussed more fully in Section IV.

III. THE VIRIAL THEOREM

In the absence of density fluctuations, the peculiar velocity of every particle in a cosmic distribution would decay like $R^{-1}(t)$. It follows that we can, in general, attribute the peculiar velocities of particles in a cosmic distribution to the action of local gravitational fields; the influence of initial conditions—if it is meaningful to speak of initial conditions in this context—quickly dies out. Let us express this idea mathematically. By integration, we obtain from the equations of motions (18)

$$v = -\int_{t_0}^{t} \frac{R(t')}{R(t)} \frac{\partial \varphi(x, t')}{\partial x} dt' + v_0 . \tag{33}$$

We now assume that t_0 has been so chosen that no appreciable correlation exists between the peculiar velocities of the same particle at times t and t_0. Forming the scalar product of equation (33) with v and taking the mass average, we then have

$$\langle v^2 \rangle_m = -\int_{t_0}^{t} \left\langle v(x, t) \cdot \frac{R(t')}{R(t)} \frac{\partial \varphi(x, t')}{\partial x} \right\rangle_m dt' . \tag{34}$$

In order to evaluate the integral on the right we shall first make a very restrictive assumption about the time-dependent behavior of local irregularities. Afterward we shall replace this assumption by a weaker one.

We shall say that the evolution of local irregularities is a *quasi-stationary* process if

$$\left\langle v(x, t) \frac{R(t')}{R(t)} \frac{\partial \varphi(x, t')}{\partial x} \right\rangle_m = F(t - t') , \tag{35}$$

where F is any smooth function of its argument. Assuming for the moment that this condition is fulfilled, we can make the following change of variables in the integrand on the right side of equation (34):

$$t' \rightarrow t_1 , \qquad t \rightarrow t_1 + \tau \qquad (\tau \equiv t - t') . \tag{36}$$

The integral in question then becomes

$$\begin{aligned} &+\int_{t-t_0}^{0} \left\langle v(x, t_1 + \tau) \frac{R(t_1)}{R(t_1 + \tau)} \cdot \frac{\partial \varphi(x, t_1)}{\partial x} \right\rangle_m d\tau \\ &= \left\langle \frac{\partial \varphi(x, t_1)}{\partial x} \cdot \int_{t-t_0}^{0} v(x, t_1 + \tau) \frac{R(t_1)}{R(t_1 + \tau)} d\tau \right\rangle_m . \end{aligned} \tag{37}$$

Now the displacement of a particle from its initial position is given in terms of its peculiar velocity by

$$x(x_0, t) - x_0 = \int_{t_0}^{t} v(x, t') \frac{R(t)}{R(t')} dt' ; \tag{38}$$

the right side of this formula represents the resultant of all the infinitesimal displacements $\boldsymbol{v}(\boldsymbol{x}, t')dt'$ suffered by the particle, each corrected for the expansion of the substratum in the interval (t', t). Thus the term (37) reduces to

$$\left\langle \frac{\partial \varphi(x_0, t_1)}{\partial x_0} \cdot [x_0 - x(x_0, t_1 + t - t_0)] \right\rangle_m = \left\langle \frac{\partial \varphi(x, t_1)}{\partial x} \cdot x \right\rangle_m . \tag{39}$$

This expression seems to depend on the choice of a co-ordinate origin but is, in fact, invariant under translation. By a short series of standard transformations (see, e.g., Chandrasekhar 1961, p. 578), we finally obtain

$$\left\langle \frac{\partial \varphi(x, t_1)}{\partial x} \cdot x \right\rangle_m = -\tfrac{1}{2} \langle \varphi(x, t_1) \rangle_m . \tag{40}$$

This is the value of the right side of equation (34) under the assumption of quasi-stationarity. Setting $t_1 = t$, we therefore obtain from equation (34) the relation

$$\langle v^2 \rangle_m + \tfrac{1}{2} \langle \varphi \rangle_m = 2T_m + U_m = 0 . \tag{41}$$

Since t_1 is arbitrary, it also follows that T_m and U_m are individually constant. Hence $(d/dt)\,(T_m + U_m)$ must vanish with $(2T_m + U_m)$. This result is consistent with the form of the energy equation (27). Conversely, the constancy of T_m and U_m follows from the energy equation (27) and the virial theorem (41). And under quasi-stationary conditions the energy equation reduces to

$$\frac{d}{dt}(T_m + U_m) = 0 . \tag{42}$$

We now return to the quasi-stationarity postulate (35), which we replace by the quite general formula

$$\left\langle v(x, t) \cdot \frac{R(t')}{R(t)} \frac{\partial \varphi(x, t')}{\partial x} \right\rangle_m = F(t', t - t') . \tag{43}$$

The postulate of quasi-stationarity asserts that the function $F(t, \tau)$ does not depend on the argument t. That this postulate cannot be fulfilled exactly in all circumstances is shown by the example of a universe consisting entirely of compact, essentially non-interacting clusters. Eventually the individual clusters will satisfy the classical virial theorem. Such a distribution is obviously not quasi-stationary, nor does it satisfy the cosmologic virial theorem (41). For example, the peculiar kinetic energy T of a stationary cluster increases monotonically with time, asymptotically approaching the kinetic energy as reckoned in the usual center-of-mass system.

A weaker assumption than quasi-stationarity is the following: We assume that over the range of values of t' for which the function $F(t', t - t')$ differs appreciably from zero it can be adequately represented by the first-order formula

$$F(t', t - t') = F(t_1, t - t') + (t' - t_1)G(t, t - t') , \tag{44}$$

where

$$G(t, \tau) \equiv \frac{\partial F(t, \tau)}{\partial t} , \tag{45}$$

and

$$t - t_1 = \frac{\int_0^\infty \tau G(t, \tau)\, d\tau}{\int_0^\infty G(t, \tau)\, d\tau} \equiv \epsilon(t) . \tag{46}$$

The essential content of this assumption is that the characteristic time associated with the variation of $F(t, \tau)$ as a function of its first argument is much greater than the characteristic time associated with its variation as a function of its second argument. Quasi-stationarity corresponds to the limiting case when the ratio between the first and second characteristic times is infinite.

Integrating equation (44), we obtain, with the help of equation (46),

$$-\int_{t_0}^{t} F(t', t-t')\,dt = \int_{-t_0}^{0} F(t_1, \tau)\,d\tau = -U_m(t_1) - U_m(t-\epsilon). \tag{47}$$

In place of equation (41), we now have

$$2T_m(t) + U_m(t-\epsilon) = 0, \tag{48}$$

which, if ϵ is sufficiently small, can be written in the form

$$2T_m + U_m = \epsilon \frac{dU_m}{dt} = -2\epsilon \frac{dT_m}{dt}. \tag{49}$$

Inserting this equation in the energy equation (27), we have

$$\frac{d}{dt}(T_m + U_m) = -\eta \frac{dU_m}{dt} = 2\eta \frac{dT_m}{dt}, \tag{50}$$

$$\eta \equiv \epsilon H. \tag{51}$$

Equation (50) has the alternative forms

$$\frac{d}{dt}[T_m + (1+\eta) U_m] = U_m \frac{d\eta}{dt}, \tag{52}$$

$$\frac{d}{dt}[(1-2\eta)T_m + U_m] = -2T_m \frac{d\eta}{dt}. \tag{53}$$

If the function $\eta(t)$ is given, equations (49) and (50), together with initial values of T_m and U_m, determine T_m and U_m for all time. The determination of $\eta(t)$ requires a more detailed consideration of the dynamics of local irregularities than we have given here. All that can be said at present is that the variations of T_m and U_m over periods of the order of H^{-1} will be small if η is small. The last condition seems very plausible, but a quantitative investigation of it would be desirable.

The preceding discussion makes no allowance for non-gravitational forces or for the effects of radiation, both of which are likely to be important only during early stages of the expansion, when it is also likely that the cosmic pressure will depart appreciably from zero. Conversely, it may be a good approximation to assume that the specific energy is constant and that the cosmic pressure is zero during those stages of the expansion when non-gravitational forces and the effects of radiation are negligible. During these stages the evolution of local irregularities does not influence the rate of cosmic expansion, which is described by the classical Einstein-Friedmann theory. Note that allowing for the internal structure of the "particles" that make up the cosmic distribution in no way alters this conclusion: since the "particles" are assumed to be in equilibrium, their internal motion and internal forces make no net contribution to the cosmic pressure.

Finally, let us consider briefly the problem of verifying the cosmologic virial theorem observationally. The chief difficulty here stems from the lack of an adequate distance criterion for external galaxies. Field galaxies are normally selected for radial-velocity measurements on the basis of apparent magnitude. In order to estimate the peculiar kinetic energy T_m, one needs to be able to separate the dispersion of the measured radial velocities about the mean (m, log z) relation into a component due to the dispersion of absolute magnitudes and a component due to the dispersion of peculiar velocities. It may perhaps be possible to effect such a separation by exploiting the fact that the two components vary differently with apparent magnitude: the dispersion in absolute magnitude results in a constant dispersion in log z (since $\Delta M \propto \Delta \log r \propto \Delta \log z$), while the dispersion in peculiar velocity—the effect we wish to measure—results in a constant dispersion in the redshift itself. The lack of adequate depth resolution also introduces considerable uncertainty into the estimation of the peculiar gravitational energy U_m from galaxy counts to a fixed limiting magnitude.

IV. DISCUSSION

The equations of motion for a test particle in a statistically homogeneous and isotropic distribution of matter can be written in two equivalent forms:

$$\frac{dV}{dt} = -\frac{4\pi}{3} G \bar{\rho} x - \frac{\partial \varphi}{\partial x} \qquad \text{(Newtonian form)} \cdot \tag{54}$$

$$\frac{d}{dt}(Rv) = -R\left(\frac{\partial \varphi}{\partial x}\right) \qquad \text{(quasi-Newtonian form)} . \tag{55}$$

In a universe composed of compact, effectively isolated, self-gravitating systems whose centers of mass are locally at rest, equation (54) reduces to the conventional statement of Newton's Second Law if the co-ordinate origin is taken to coincide with the center of mass of the system containing the test particle under consideration. More generally, the right side of equation (54) represents the force per unit mass exerted on a test particle by the matter contained in a sufficiently large sphere centered on the origin. One must bear in mind, however, that the first term on the right side of equation (54) does not represent the force exerted by the uniform component of the *actual* distribution; this force vanishes everywhere, by symmetry. The term in question results from the non-inertial character of the co-ordinate system, which becomes increasingly pronounced with increasing distance from the origin. Thus the Newtonian form of equation (54) masks its essentially non-Newtonian character.[7]

The non-Newtonian character of the motion is clearly shown by equation (55), which relates the rate of change of the velocity of a particle relative to the local standard of rest to the gravitational force acting on the particle. For a uniform cosmic distribution ($\sigma \equiv \varphi \equiv 0$) we have the following analogue of Newton's First Law of Motion: In the absence of local irregularities the velocity of every test particle varies as $R^{-1}(t)$.

The quasi-Newtonian energy equation,

$$\frac{d}{dt}[\tfrac{1}{2}\langle v^2 \rangle_m + \tfrac{1}{2}\langle \varphi \rangle_m] + H[\langle v^2 \rangle_m + \tfrac{1}{2}\langle \varphi \rangle_m] = 0 , \tag{56}$$

shows that conservation of energy does not, in general, obtain in a cosmic distribution, though, by virtue of the cosmologic virial theorem,

$$\langle v^2(t) \rangle_m + \tfrac{1}{2}\langle \varphi(t-\epsilon) \rangle_m = 0 , \tag{57}$$

energy may be approximately conserved under certain conditions.

The evolution of local irregularities affects the rate of cosmic expansion through the cosmic pressure, given by

$$3\bar{p} = \bar{\rho}[\langle v^2 \rangle_m + \tfrac{1}{2}\langle \varphi \rangle_m] . \tag{58}$$

[7] For a fuller discussion of this point, which has frequently been misunderstood, the reader may consult two notes by Layzer (1954*b*, 1956*b*).

When energy is conserved, the cosmic pressure vanishes, and the expansion is unaffected by the evolution of local irregularities.

The considerations of the present paper need to be generalized in two ways. In the first place, non-gravitational forces, as well as radiation and radiative processes, need to be considered. In the second place, we need a more detailed statistical description of the evolution of local irregularities than the energy equation and the virial theorem provide. In order to obtain such a description, it will probably be necessary to work explicitly with two-point averages, such as the density covariance $\langle\sigma(\boldsymbol{x})\sigma(\boldsymbol{x}+\boldsymbol{y})\rangle$. This entails a certain increase in the complexity of the mathematical description. As long as one works exclusively with one-point averages, the description involves only the cosmic time t. Problems in which only one-point averages figure are thus analogous to problems in Newtonian physics pertaining to spherically symmetric systems in a steady state. The introduction of two-point averages increases the number of independent variables to two. Fortunately, three-or-more-point averages seem to be comparatively unimportant physically, so that it will probably not be necessary to deal with equations involving more than two independent variables.

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