A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II

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In this paper, we shall show how the theory of measurements is to be understood from the point of view of a physical interpretation of the quantum theory in terms of "hidden" variables, developed in a previous paper. We find that in principle, these "hidden" variables determine the precise results of each individual measurement process. In practice, however, in measurements that we now know how to carry out, the observing apparatus disturbs the observed system in an unpredictable and uncontrollable way, so that the uncertainty principle is obtained as a practical limitation on the possible precision of measurements. This limitation is not, however, inherent in the conceptual structure of our interpretation. We shall see, for example, that simultaneous measurements of position and momentum having unlimited precision would in principle be possible if, as suggested in the previous paper, the mathematical formulation of the quantum theory needs to be modified at very short distances in certain ways that are consistent with our interpretation but not with the usual interpretation.

We give a simple explanation of the origin of quantum-mechanical correlations of distant objects in the hypothetical experiment of Einstein, Podolsky, and Rosen, which was suggested by these authors as a criticism of the usual interpretation.

Finally, we show that von Neumann's proof that quantum theory is not consistent with hidden variables does not apply to our interpretation, because the hidden variables contemplated here depend both on the state of the measuring apparatus and the observed system and therefore go beyond certain of von Neumann's assumptions.

In two appendixes, we treat the problem of the electromagnetic field in our interpretation and answer certain additional objections which have arisen in the attempt to give a precise description for an individual system at the quantum level.

1. INTRODUCTION

In a previous paper,[1] to which we shall hereafter refer as I, we have suggested an interpretation of the quantum theory in terms of "hidden" variables. We have shown that although this interpretation provides a conceptual framework that is broader than that of the usual interpretation, it permits of a set of three mutually consistent special assumptions, which lead to the same physical results as are obtained from the usual interpretation of the quantum theory. These three special assumptions are: (1) The ψ-field satisfies Schroedinger's equation. (2) If we write $\psi = R\exp(is/\hbar)$, then the particle momentum is restricted to $\mathbf{p} = \nabla S(\mathbf{x})$. (3) We have a statistical ensemble of particle positions, with a probability density, $P = |\psi(\mathbf{x})|^2$. If the above three special assumptions are not made, then one obtains a more general theory that cannot be made consistent with the usual interpretation. It was suggested in Paper I that such generalizations may actually be needed for an understanding of phenomena associated with distances of the order of 10^{-13} cm or less, but may produce changes of negligible importance in the atomic domain.

In this paper, we shall apply the interpretation of the quantum theory suggested in Paper I to the development of a theory of measurements in order to show that as long as one makes the special assumptions indicated above, one is led to the same predictions for all measurements as are obtained from the usual interpretation. In our interpretation, however, the uncertainty principle is regarded, not as an inherent limitation on the precision with which we can correctly conceive of the simultaneous definition of momentum and position, but rather as a practical limitation on the precision with which these quantities can simultaneously be measured, arising from unpredictable and uncontrollable disturbances of the observed system by the measuring apparatus. If the theory needs to be generalized in the ways suggested in Paper I, Secs. 4 and 9, however, then these disturbances could in principle either be eliminated, or else be made subject to prediction and control, so that their effects could be corrected for. Our interpretation therefore demonstrates that measurements violating the uncertainty principle are at least conceivable.

2. QUANTUM THEORY OF MEASUREMENTS

We shall now show how the quantum theory of measurements is to be expressed in terms of our suggested interpretation of the quantum theory.[2]

In general, a measurement of any variable must always be carried out by means of an interaction of the system of interest with a suitable piece of measuring apparatus. The apparatus must be so constructed that any given state of the system of interest will lead to a certain range of states of the apparatus. Thus, the interaction introduces correlations between the state of the observed system and the state of the apparatus. The range of indefiniteness in this correlation may be called the uncertainty, or the error, in the measurement.

Let us now consider an observation designed to measure an arbitrary (hermitian) "observable" Q, associated with an electron. Let $\mathbf{x}$ represent the position of the electron, y that of the significant apparatus coordinate (or coordinates if there are more than one). Now, one can show[2] that it is enough to consider an impulsive measurement, i.e., a measurement utilizing a very strong interaction between apparatus and system

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[1] D. Bohm, Phys. Rev. 84, 166 (1951).

[2] For a treatment of how the theory of measurements can be carried out with the usual interpretation, see D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., New York, 1951), Chapter 22.

under observation, which lasts for so short a time that the changes of the apparatus and the system under observation that would have taken place in the absence of interaction can be neglected. Thus, at least while the interaction is taking place, we can neglect the parts of the Hamiltonian associated with the apparatus alone and with the observed system alone, and we need retain only the part of the Hamiltonian, H_I, representing the interaction. Moreover, if the Hamiltonian operator is chosen to be a function only of quantities that commute with Q, then the interaction process will produce no uncontrollable changes in the observable, Q, but only in observables that do not commute with Q. In order that the apparatus and the system under observation shall be coupled, however, it is necessary that H_I shall also depend on operators involving y.

For the sake of illustration of the principles involved, we shall consider the following interaction Hamiltonian:

$$H_I = -aQp_y, \tag{1}$$

where a is a suitable constant and p_y is the momentum conjugate to y.

Now, in our interpretation, the system is to be described by a four-dimensional but objectively real wave field that is a function of $\mathbf{x}$ and y and by a corresponding four-dimensional representative point, specified by the coordinates, $\mathbf{x}$, of the electron and the coordinate, y, of the apparatus. Since the motion of the representative point is in part determined by forces produced by the ψ-field acting on both electron and apparatus variables, our first step in solving this problem is to calculate the ψ-field. This is done by solving Schroedinger's equation, with the appropriate boundary conditions on ψ.

Now, during interaction, Schroedinger's equation is approximated by

$$i\hbar\partial\Psi/\partial t = -aQp_y\Psi = (ia/\hbar)Q\partial\Psi/\partial y. \tag{2}$$

It is now convenient to expand Ψ in terms of the complete set $\psi_q(\mathbf{x})$ of eigenfunctions of the operator, Q, where q denotes an eigenvalue of Q. For the sake of simplicity, we assume that the spectrum of Q is discrete, although the results are easily generalized to a continuous spectrum. Denoting the expansion coefficients by $f_q(y, t)$, we obtain

$$\Psi(\mathbf{x}, y, t) = \textstyle\sum_q \psi_q(\mathbf{x}) f_q(y, t). \tag{3}$$

Noting that $Q\psi_q(\mathbf{x}) = q\psi_q(\mathbf{x})$, we readily verify that Eq. (2) can now be reduced to the following series of equations for $f_q(y, t)$:

$$i\hbar\partial f_q(y, t)/\partial t = (ia/\hbar)qf_q(y, t). \tag{4}$$

If the initial value of $f_q(y, t)$ was $f_q{}^0(y)$, we obtain as a solution

$$f_q(y, t) = f_q{}^0(y - aqt/\hbar^2), \tag{5}$$

and

$$\Psi(\mathbf{x}, y, t) = \textstyle\sum_q \psi_q(\mathbf{x}) f_q{}^0(y - aqt/\hbar^2). \tag{6}$$

Now, initially the apparatus and the electron were independent. As shown in Paper I, Sec. 7, in our interpretation (as in the usual interpretation), independent systems must have wave fields $\Psi(\mathbf{x}, y, t)$ that are equal to a product of a function of $\mathbf{x}$ and a function of y. Initially, we therefore have

$$\Psi_0(\mathbf{x}, y) = \psi_0(\mathbf{x})g_0(y) = g_0(y)\textstyle\sum_q c_q\psi_q(\mathbf{x}), \tag{7}$$

where the c_q are the (unknown) expansion coefficients of $\psi_q(\mathbf{x})$, and $g_0(y)$ is the initial wave function of the apparatus coordinate, y. The function $g_0(y)$ will take the form of a packet. For the sake of convenience, we assume that this packet is centered at $y=0$ and that its width is Δy. Normally, because the apparatus is classically describable, the definition of this packet is far less precise than that allowed by the limits of precision set by the uncertainty principle.

From Eqs. (7) and (3), we shall readily deduce that $f_q{}^0(y) = c_q g_0(y)$. When this value of $f_q{}^0(y)$ is inserted into Eq. (6), we obtain

$$\Psi(\mathbf{x}, y, t) = \textstyle\sum_q c_q\psi_q(\mathbf{x})g_0(y - aqt/\hbar^2). \tag{8}$$

Equation (8) indicates already that the interaction has introduced a correlation between q and the apparatus coordinate, y. In order to show what this correlation means in our interpretation of the quantum theory, we shall use some arguments that have been developed in more detail in Paper I, Sec. 7, in connection with a similar problem involving the interaction of two particles in a scattering process. First we note that while the electron and the apparatus are interacting, the wave function (8) becomes very complicated, so that if it is expressed as

$$\Psi(\mathbf{x}, y, t) = R(\mathbf{x}, y, t)\exp[iS(\mathbf{x}, y, t)/\hbar],$$

then R and S undergo rapid oscillations both as a function of position and of time. From this we deduce that the "quantum-mechanical" potential,

$$U = (-\hbar^2/2mR)(\nabla_x{}^2R + \partial^2R/\partial y^2),$$

undergoes violent fluctuations, especially where R is small, and that the particle momenta, $\mathbf{p} = \nabla_x S(\mathbf{x}, y, t)$ and $p_y = \partial S(\mathbf{x}, y, t)/\partial y$, also undergo corresponding violent and extremely complicated fluctuations. Eventually, however, if the interaction continues long enough, the behavior of the system will become simpler because the packets $g_0(y - aqt/\hbar^2)$, corresponding to different values of q, will cease to overlap in y space. To prove this, we note that the center of the qth packet in y space is at

$$y = aqt/\hbar^2; \quad \text{or} \quad q = \hbar^2 y/at. \tag{9}$$

If we denote the separation of adjacent values of q by δq, we then obtain for the separation of the centers of adjacent packets in y space

$$\delta y = at\delta q/\hbar^2. \tag{10}$$

It is clear that if the product of the strength of interaction a, and the duration of interaction, t, is large enough, then δy can be made much larger than the

width Δy of the packet. Then packets corresponding to different values of q will cease to overlap in y space and will, in fact, obtain separations large enough to be classically describable.

Because the probability density is equal to $|\Psi|^2$, we deduce that the apparatus variable, y, must finally enter one of the packets and remain with that packet thereafter (since it does not enter the intermediate space between packets in which the probability density is practically zero). Now, the packet entered by the apparatus variable y determines the actual result of the measurement, which the observer will obtain when he looks at the apparatus. The other packets can (as shown in Paper I, Sec. 7) be ignored, because they affect neither the quantum-mechanical potential acting on the particle coordinates $\mathbf{x}$ and y, nor the particle momenta, $\mathbf{p}_x = \nabla_x S$ and $p_y = \partial S/\partial y$. Moreover, the wave function can also be renormalized without affecting any of the above quantities. Thus, for all practical purposes, we can replace the complete wave function, Eq. (8), by a new renormalized wave function

$$\Psi(\mathbf{x}, y) = \psi_q(\mathbf{x}) g_0(y - aqt/\hbar^2), \tag{11}$$

where q now corresponds to the packet actually containing the apparatus variable, y. From this wave function, we can deduce, as shown in Paper I, Sec. 7, that the apparatus and the electron will subsequently behave independently. Moreover, by observing the approximate value of the apparatus coordinate within an error $\Delta y \ll \delta y$, we can deduce with the aid of Eq. (9) that since the electron wave function can for all practical purposes be regarded as $\psi_q(\mathbf{x})$, the observable, Q, must have the definite value, q. However, if the product, "$at\delta q/\hbar^2$," appearing in Eqs. (8), (9), (10), and (11), had been less than Δy, then no clear measurement of Q would have been possible, because packets corresponding to different q would have overlapped, and the measurement would not have had the requisite accuracy.[3]

Finally, we note that even if the apparatus packets are subsequently caused to overlap, none of those conclusions will be altered. For the apparatus variable y will inevitably be coupled to a whole host of internal thermodynamic degrees of freedom, $y_1, y_2, \cdots y_s$, as a result of effects such as friction and brownian motion. As shown in Paper I, Sec. 7, interference between packets corresponding to different values of q would be possible only if the packets overlapped in the space of $y_1, y_2, \cdots y_s$, as well as in y space. Such an overlap, however, is so improbable that for all practical purposes, we can ignore the possibility that it will ever occur.

3. THE ROLE OF PROBABILITY IN MEASUREMENTS—THE UNCERTAINTY PRINCIPLE

In principle, the final result of a measurement is determined by the initial form of the wave function of the combined system, $\Psi^0(\mathbf{x}, y)$, and by the initial position of the electron particle, $\mathbf{x}_0$, and the apparatus variable, y_0. In practice, however, as we have seen, the orbit fluctuates violently while interaction takes place, and is very sensitive to the precise initial values of $\mathbf{x}$ and y, which we can neither predict nor control. All that we can predict in practice is that in an ensemble of similar experiments performed under equivalent initial conditions, the probability density is $|\Psi(\mathbf{x}, y)|^2$. From this information, however, we are able to calculate only the probability that in an individual experiment, the result of a measurement of Q will be a specific number q. To obtain the probability of a given value of q, we need only integrate the above probability density over all $\mathbf{x}$ and over all values of y in the neighborhood of the qth packet. Because the packets do not overlap, the Ψ-field in this region is equal to $c_q\psi_q(\mathbf{x})g_0(y - aqt/\hbar^2)$ [see Eq. (8)]. Since, by definition, $\psi_q(\mathbf{x})$ and $g_0(y)$ are normalized, the total probability that a particle is in the qth packet is

$$P_q = |c_q|^2. \tag{12}$$

The above is, however, just what is obtained from the usual interpretation. We conclude then that our interpretation is capable of leading in all possible experiments to identical predictions with those obtained from the usual interpretation (provided, or course, that we make the special assumptions indicated in the introduction).

Let us now see what a measurement of the observable, Q, implies with regard to the state of the electron particle and its Ψ-field. First, we note that the process of interaction with an apparatus designed to measure the observable, Q, effectively transforms the electron ψ-field from whatever it was before the measurement took place into an eigenfunction $\psi_q(\mathrm{x})$ of the operator Q. The precise value of q that comes out of this process is as we have seen, not, in general, completely predictable or controllable. If, however, the same measurement is repeated after the ψ-field has been transformed into $\psi_q(\mathbf{x})$, we can then predict that (as in the usual interpretation), the same value of q, and therefore the same wave function, $\psi_q(\mathbf{x})$, will be obtained again. If, however, we measure an observable "P" that does not commute with Q, then the results of this measurement are not, in practice, predictable or controllable. For as shown in Eq. (8), the Ψ-field after interaction with the measuring apparatus is now transformed into

$$\Psi(\mathbf{x}, z, t) = \textstyle\sum_p a_{p,q}\phi_p(\mathbf{x}) g_0(z - apt/\hbar^2), \tag{13}$$

where $\phi_p(\mathbf{x})$ is an eigenfunction of the operator, P, belonging to an eigenvalue, p, and where $a_{p,q}$ is an expansion coefficient defined by

$$\psi_q(\mathbf{x}) = \textstyle\sum_p a_{p,q}\phi_p(\mathbf{x}). \tag{14}$$

Since the packets corresponding to different p ultimately become completely separate in z space, we deduce, as in the case of the measurement of Q, that

[3] A similar requirement is obtained in the usual interpretation. See reference 2, Chapter 22, Sec. 8.

for all practical purposes, this wave function may be replaced by

$$\Psi = a_{pq}\phi_p(\mathbf{x})g_0(z - apt/\hbar^2),$$

where p now represents the packet actually entered by the apparatus coordinate, y. As in the case of measurement of Q, we readily show that the precise value of p that comes out of this experiment cannot be predicted or controlled and that the probability of a given value of p is equal to $|a_{pq}|^2$. This is, however, just what is obtained in the usual interpretation of this process.

It is clear that if two "observables," P and Q, do not commute, one cannot carry out a measurement of both simultaneously on the same system. The reason is that each measurement disturbs the system in a way that is incompatible with carrying out the process necessary for the measurement of the other. Thus, a measurement of P requires that wave field, ψ, shall become an eigenfunction of P, while a measurement of Q requires that it shall become an eigenfunction of Q. If P and Q do not commute, then by definition, no ψ-function can be simultaneously an eigenfunction of both. In this way, we understand in our interpretation why measurements, of complementary quantities, must (as in the usual interpretation) necessarily be limited in their precision by the uncertainty principle.

4. PARTICLE POSITIONS AND MOMENTA AS "HIDDEN VARIABLES"

We have seen that in measurements that can now be carried out, we cannot make precise inferences about the particle position, but can say only that the particle must be somewhere in the region in which $|\psi|$ is appreciable. Similarly, the momentum of a particle that happens to be at the point, $\mathbf{x}$, is given by $\mathbf{p} = \nabla S(\mathbf{x})$, so that since $\mathbf{x}$ is not known, the precise value of $\mathbf{p}$ is also not, in general, inferrable. Hence, as long as we are restricted to making observations of this kind, the precise values of the particle position and momentum must, in general, be regarded as "hidden," since we cannot at present measure them. They are, however, connected with real and already observable properties of matter because (along with the ψ-field) they determine in principle the actual result of each individual measurement. By way of contrast, we recall here that in the usual interpretation of the theory, it is stated that although each measurement admittedly leads to a definite number, nothing determines the actual value of this number. The result of each measurement is assumed to arise somehow in an inherently indescribable way that is not subject to a detailed analysis. Only the statistical results are said to be predictable. In our interpretation, however, we assert that the at present "hidden" precisely definable particle positions and momenta determine the results of each individual measurement process, but in a way whose precise details are so complicated and uncontrollable, and so little known, that one must for all practical purposes restrict oneself to a statistical description of the connection between the values of these variables and the directly observable results of measurements. Thus, we are unable at present to obtain direct experimental evidence for the existence of precisely definable particle positions and momenta.

5. "OBSERVABLES" OF USUAL INTERPRETATION ARE NOT A COMPLETE DESCRIPTION OF SYSTEM IN OUR INTERPRETATION

We have seen in Sec. 3 that in the measurement of an "observable," Q, we cannot obtain enough information to provide a complete specification of the state of an electron, because we cannot infer the precisely defined values of the particle momentum and position, which are, for example, needed if we wish to make precise predictions about the future behavior of the electron. Moreover, the process of measuring an observable does not provide any unambiguous information about the state that existed before the measurement took place; for in such a measurement, the ψ-field is transformed into an in practice unpredictable and uncontrollable eigenfunction, $\psi_q(\mathbf{x})$, of the measured "observable" Q. This means that the measurement of an "observable" is not really a measurement of any physical property belonging to the observed system alone. Instead, the value of an "observable" measures only an incompletely predictable and controllable potentiality belonging just as much to the measuring apparatus as to the observed system itself.[4] At best, such a measurement provides unambiguous information only at a classical level of accuracy, where the disturbance of the ψ-field by the measuring apparatus can be neglected. The usual "observables" are therefore not what we ought to try to measure at a quantum level of accuracy. In Sec. 6, we shall see that it is conceivable that we may be able to carry out new kinds of measurements, providing information not about "observables" having a very ambiguous significance, but rather about physically significant properties of a system, such as the actual values of the particle position and momentum.

As an example of the rather indirect and ambiguous significance of the "observable," we may consider the problem of measuring the momentum of an electron. Now, in the usual interpretation, it is stated that one can always measure the momentum "observable" without changing the value of the momentum. The result is said, for example, to be obtainable with the aid of an impulsive interaction involving only operators which commute with the momentum operator, $\mathbf{p}_x$. To represent such a measurement, we could choose $H_I = -ap_xp_y$ in Eq. (1). In our interpretation, however, we cannot in general conclude that such an interaction will enable us to measure the actual particle momentum without changing its value. In fact, in our interpreta-

[4] Even in the usual interpretation, an observation must be regarded as yielding a measure of such a potentiality. See reference 2, Chapter 6, Sec. 9.

tion, a measurement of particle momentum that does not change the value of this momentum is possible only if the ψ-field initially takes the special form, $\exp(i\mathbf{p}\cdot\mathbf{x}/\hbar)$. If, however, ψ initially takes its most general possible form,

$$\psi=\sum_p a_1\mathbf{p}\exp(i\mathbf{p}\cdot\mathbf{x}/\hbar), \tag{15}$$

then as we have seen in Secs. 2 and 3, the process of measuring the "observable" p_x will effectively transform the ψ-field of the electron into

$$\exp(ip_x/\hbar) \tag{16}$$

with a probability $|a_p|^2$ that a given value of p_x will be obtained. When the ψ-field is altered in this way, large quantities of momentum can be transferred to the particle by the changing ψ-field, even though the interaction Hamiltonian, H_I, commutes with the momentum operator, $\mathbf{p}$.

As an example, we may consider a stationary state of an atom, of zero angular momentum. As shown in Paper I, Sec. 5, the ψ-field for such a state is real, so that we obtain

$$\mathbf{p}=\nabla S=0.$$

Thus, the particle is at rest. Nevertheless, we see from Eqs. (14) and (15) that if the momentum "observable" is measured, a large value of this "observable" may be obtained if the ψ-field happens to have a large fourier coefficient, $a_\mathbf{p}$, for a high value of $\mathbf{p}$. The reason is that in the process of interaction with the measuring apparatus, the ψ-field is altered in such a way that it can give the electron particle a correspondingly large momentum, thus transforming some of the potential energy of interaction of the particle with its ψ-field into kinetic energy.

A more striking illustration of the points discussed above is afforded by the problem of a "free" particle contained between two impenetrable and perfectly reflecting walls, separated by a distance L. For this case, the spatial part of the ψ-field is

$$\psi=\sin(2\pi nx/L),$$

where n is an integer and the energy of the electron is

$$E=(1/2m)(nh/L)^2.$$

Because the ψ-field is real, we deduce that the particle is at rest.

Now, at first sight, it may seem puzzling that a particle having a high energy should be at rest in the empty space between two walls. Let us recall, however, that the space is not really empty, but contains an objectively real ψ-field that can act on the particle. Such an action is analogous to (but of course not identical with) the action of an electromagnetic field, which could create non-uniform motion of the particle in this apparently "empty" enclosure. We observe that in our problem, the ψ-field is able to bring the particle to rest and to transform the entire kinetic energy into potential energy of interaction with the ψ-field. To prove this, we evaluate the "quantum-mechanical potential" for this ψ-field

$$U=\frac{-\hbar^2}{2m}\frac{\nabla^2 R}{R}=\frac{-\hbar^2}{2m}\frac{\nabla^2\psi}{\psi}=\frac{1}{2m}\left(\frac{nh}{L}\right)^2$$

and note that it is precisely equal to the total energy, E.

Now, as we have seen, any measurement of the momentum "observable" must change the ψ-field in such a way that in general some (and in our case, all) of this potential energy is transformed into kinetic energy. We may use as an illustration of this general result a very simple specific method of measuring the momentum "observable," namely, to remove the confining walls suddenly and then to measure the distance moved by the particle after a fairly long time. We can compute the momentum by dividing this distance by the time of transit. If (as in the usual interpretation of the quantum theory) we assume that the electron is "free," then we conclude that the process of removing the walls should not appreciably change the momentum if we do it fast enough, for the probability that the particle is near a wall when this happens can then in principle be made arbitrarily small. In our interpretation, however, the removal of the walls alters the particle momentum indirectly, because of its effect on the ψ-field, which acts on the particle. Thus, after the walls are removed, two wave packets moving in opposite directions begin to form, and ultimately they become completely separate in space. Because the probability density is $|\psi|^2$, we deduce that the particle must end up in one packet or the other. Moreover, the reader will readily convince himself that the particle momentum will be very close to $\pm nh/L$, the sign depending on which packet the particle actually enters. As in Sec. (2), the packet not containing the particle can subsequently be ignored. In principle, the final particle momentum is determined by the initial form of the ψ-field and by the initial particle position. Since we do not in practice know the latter, we can at best predict a probability of $\frac{1}{2}$ that the particle ends up in either packet. We conclude then that this measurement of the momentum "observable" leads to the same result as is predicted in the usual interpretation. However, the actual particle momentum existing before the measurement took place is quite different from the numerical value obtained for the momentum "observable," which, in the usual interpretation, is called the "momentum."

6. ON THE POSSIBILITY OF MEASUREMENTS OF UNLIMITED PRECISION

We have seen that the so-called "observables" do not measure any very readily interpretable properties of a system. For example, the momentum "observable" has in general no simple relation to the actual particle momentum. It may therefore be fruitful to consider

how we might try to measure properties which, according to our interpretation, are (along with the ψ-field) the physically significant properties of an electron, namely, the actual particle position and momentum. In connection with this problem, we shall show that if, as suggested in Paper I, Secs. 4 and 9, we give up the three mutually consistent special assumptions leading to the same results as those of the usual interpretation of the quantum theory, then in our interpretation, the particle position and momentum can in principle be measured simultaneously with unlimited precision.

Now, for our purposes, it will be adequate to show that precise predictions of the future behavior of a system are in principle possible. In our interpretation, a sufficient condition for precise predictions is as we have seen that we shall be able to prepare a system in a state in which the ψ-field and the initial particle position and momentum are precisely known. We have shown that it is possible, by measuring the "observable," Q, with the aid of methods that are now available, to prepare a state in which the ψ-field is effectively transformed into a known form, $\psi_q(\mathbf{x})$; but we cannot in general predict or control the precise position and momentum of the particle. If we could now measure the position and momentum of the particle without altering the ψ-field, then precise predictions would be possible. However, the results of Secs. 2, 3, and 4 prove that as long as the three special assumptions indicated above are valid, we cannot measure the particle position more accurately without effectively transforming the ψ-function into an incompletely predictable and controllable packet that is much more localized than $\psi_q(\mathbf{x})$. Thus, efforts to obtain more precise definition of the state of the system will be defeated. But it is clear that the difficulty originates in the circumstance that the potential energy of interaction between electron and apparatus, $V(\mathbf{x}, y)$, plays two roles. For it not only introduces a direct interaction between the two particles, proportional in strength to $V(\mathbf{x}, y)$ itself, but it introduces an indirect interaction between these particles, because this potential also appears in the equation governing the ψ-field. This indirect interaction may involve rapid and violent fluctuations, even when $V(\mathbf{x}, y)$ is small. Thus, we are led to lose control of the effects of this interaction, because no matter how small $V(\mathbf{x}, y)$ is, very large and chaotically complicated disturbances in the particle motion may occur.

If, however, we give up the three special assumptions mentioned previously, then it is not inherent in our conceptual structure that every interaction between particles must inevitably also produce large and uncontrollable changes in the ψ-field. Thus, in Paper I, Eq. (31), we give an example in which we postulate a force acting on a particle that is not necessarily accompanied by a corresponding change in the ψ-field. Equation (31), Paper I, is concerned only with a one-particle system, but similar assumptions can be made for systems of two or more particles. In the absence of any specific theory, our interpretation permits an infinite number of kinds of such modifications, which can be chosen to be important at small distances but negligible in the atomic domain. For the sake of illustration, suppose that it should turn out that in certain processes connected with very small distances, the force acting on the apparatus variable is

$$F_y = ax,$$

where a is a constant. Now if "a" is made large enough so that the interaction is impulsive, we can neglect all changes in y that are brought about by the forces that would have been present in the absence of this interaction. Moreover, for the sake of illustration of the principles involved, we are permitted to make the assumption, consistent with our interpretation, that the force on the electron is zero. The equation of motion of y is then

$$\ddot{y} = ax/m.$$

The solution is

$$y - y_0 = (axt^2/2m) + \dot{y}_0 t,$$

where $\dot{y}_0$ is the initial velocity of the apparatus variable and y_0 its initial position. Now, if the product, at^2, is large enough, then $y - y_0$ can be made much larger than the uncertainty in y arising from the uncertainty of y_0, and the uncertainty of $\dot{y}_0$. Thus, $y - y_0$ will be determined primarily by the particle position, x. In this way, it is conceivable that we could obtain a measurement of x that does not significantly change $x, \dot{x}$, or the ψ-function. The particle momentum can then be obtained from the relation, $p = \nabla S(\mathbf{x})$, where $S/\hbar$ is the phase of the ψ-function. Thus, precise predictions would in principle be possible.

7. THE ORIGIN OF THE STATISTICAL ENSEMBLE IN THE QUANTUM THEORY

We shall now see that even if, because of a failure of the three special assumptions mentioned in Secs. 1 and 6, we are able to determine the particle positions and momenta precisely, we shall nevertheless ultimately obtain a statistical ensemble again at the atomic level, with a probability density equal to $|\psi|^2$. The need for such an ensemble arises from the chaotically complicated character of the coupling between the electron and classical systems, such as volumes of gas, walls of containers, pieces of measuring apparatus, etc., with which this particle must inevitably in practice interact. For as we have seen in Sec. 2, and in Paper I, Sec. 7, during the course of such an interaction, the "quantum-mechanical" potential undergoes violent and rapid fluctuations, which tend to make the particle orbit wander over the whole region in which the ψ-field is appreciable. Moreover, these fluctuations are further complicated by the effects of molecular chaos in the very large number of internal thermodynamic degrees of freedom of these classically describable systems, which are inevitably excited in any interaction process.

Thus, even if the initial particle variables were well defined, we should soon in practice lose all possibility of following the particle motion and would be forced to have recourse to some kind of statistical theory. The only question that remains is to show why the probability density that ultimately comes about should be equal to $|\psi|^2$ and not to some other quantity.

To answer this question, we first note that a statistical ensemble with a probability density $|\psi(\mathbf{x})|^2$ has the property that under the action of forces which prevail at the atomic level, where our three special assumptions are satisfied, it will be preserved by the equations of motion of the particles, once it comes into existence. There remains only the problem of showing that an arbitrary deviation from this ensemble tends, under the action of the chaotically complicated forces described in the previous paragraph, to decay into an ensemble with a probability density of $|\psi(\mathbf{x})|^2$. This problem is very similar to that of proving Boltzmann's H theorem, which shows in connection with a different but analogous problem that an arbitrary ensemble tends as a result of molecular chaos to decay into an equilibrium Gibbs ensemble. We shall not carry out a detailed proof here, but we merely suggest that it seems plausible that one could along similar lines prove that in our problem, an arbitrary ensemble tends to decay into an ensemble with a density of $|\psi(\mathbf{x})|^2$. These arguments indicate that in our interpretation, quantum fluctuations and classical fluctuations (such as the Brownian motion) have basically the same origin; *viz.*, the chaotically complicated character of motion at the microscopic level.

8. THE HYPOTHETICAL EXPERIMENT OF EINSTEIN, PODOLSKY, AND ROSEN

The hypothetical experiment of Einstein, Podolsky, and Rosen[5] is based on the fact that if we have two particles, the sum of their momenta, $p=p_1+p_2$, commutes with the difference of their positions, $\xi=x_1-x_2$. We can therefore define a wave function in which p is zero, while ξ has a given value, a. Such a wave function is

$$\psi=\delta(x_1-x_2-a). \tag{17}$$

In the usual interpretation of the quantum theory, p_1-p_2 and x_1+x_2 are completely undetermined in a system having the above wave function.

The whole experiment centers on the fact that an observer has a choice of measuring either the momentum or the position of any one of the two particles. Whichever of these quantities he measures, he will be able to infer a definite value of the corresponding variable in the other particle, because of the fact that the above wave function implies correlations between variables belonging to each particle. Thus, if he obtains a position x_1 for the first particle, he can infer a position of $x_2=a-x_1$ for the second particle; but he loses all possibility of making any inferences about the momenta of either particle. On the other hand, if he measures the momentum of the first particle and obtains a value of p_1, he can infer a value of $p_2=-p_1$ for the momentum of the second particle; but he loses all possibility of making any inferences about the position of either particle. Now, Einstein, Podolsky, and Rosen believe that this result is itself probably correct, but they do not believe that quantum theory as usually interpreted can give a complete description of how these correlations are propagated. Thus, if these were classical particles, we could easily understand the propagation of correlations because each particle would then simply move with a velocity opposite to that of the other. But in the usual interpretation of quantum theory, there is no similar conceptual model showing in detail how the second particle, which is not in any way supposed to interact with the first particle, is nevertheless able to obtain either an uncontrollable disturbance of its position or an uncontrollable disturbance of its momentum depending on what kind of measurement the observer decided to carry out on the first particle. Bohr's point of view is, however, that no such model should be sought and that we should merely accept the fact that these correlations somehow manage to appear. We must note, of course, that the quantum-mechanical description of these processes will always be consistent, even though it gives us no precisely definable means of describing and analyzing the relationships between the classically describable phenomena appearing in various pieces of measuring apparatus.

In our suggested new interpretation of the quantum theory, however, we can describe this experiment in terms of a single precisely definable conceptual model, for we now describe the system in terms of a combination of a six-dimensional wave field and a precisely definable trajectory in a six-dimensional space (see Paper I, Sec. 6). If the wave function is initially equal to Eq. (17), then since the phase vanishes, the particles are both at rest. Their possible positions are, however, described by an ensemble, in which $x_1-x_2=a$. Now, if we measure the position of the first particle, we introduce uncontrollable fluctuations in the wave function for the entire system, which, through the "quantum-mechanical" forces, bring about corresponding uncontrollable fluctuations in the momentum of each particle. Similarly, if we measure the momentum of the first particle, uncontrollable fluctuations in the wave function for the system bring about, through the "quantum-mechanical" forces, corresponding uncontrollable changes in the position of each particle. Thus, the "quantum-mechanical" forces may be said to transmit uncontrollable disturbances instantaneously from one particle to another through the medium of the ψ-field.

What does this transmission of forces at an infinite rate mean? In nonrelativistic theory, it certainly causes

[5] Einstein, Podolsky, and Rosen, Phys. Rev. **47**, 777 (1933).

no difficulties. In a relativistic theory, however, the problem is more complicated. We first note that as long as the three special assumptions mentioned in Sec. 2 are valid, our interpretation can give rise to no inconsistencies with relativity, because it leads to precisely the same predictions for all physical processes as are obtained from the usual interpretation (which is known to be consistent with relativity). The reason why no contradictions with relativity arise in our interpretation despite the instantaneous transmission of momentum between particles is that no signal can be carried in this way. For such a transmission of momentum could constitute a signal only if there were some practical means of determining precisely what the second particle would have done if the first particle had not been observed; and as we have seen, this information cannot be obtained as long as the present form of the quantum theory is valid. To obtain such information, we require conditions (such as might perhaps exist in connection with distances of the order of 10^{-13} cm) under which the usual form of the quantum theory breaks down (see Sec. 6), so that the positions and momenta of the particles can be determined simultaneously and precisely. If such conditions should exist, then there are two ways in which contradictions might be avoided. First, the more general physical laws appropriate to the new domains may be such that they do not permit the transmission of controllable aspects of interparticle forces faster than light. In this way, Lorentz covariance could be preserved. Secondly, it is possible that the application of the usual criteria of Lorentz covariance may not be appropriate when the usual interpretation of quantum theory breaks down. Even in connection with gravitational theory, general relativity indicates that the limitation of speeds to the velocity of light does not necessarily hold universally. If we adopt the spirit of general relativity, which is to seek to make the properties of space dependent on the properties of the matter that moves in this space, then it is quite conceivable that the metric, and therefore the limiting velocity, may depend on the ψ-field as well as on the gravitational tensor $g^{\mu,\nu}$. In the classical limit, the dependence on the ψ-field could be neglected, and we would get the usual form of covariance. In any case, it can hardly be said that we have a solid experimental basis for requiring the same form of covariance at very short distances that we require at ordinary distances.

To sum up, we may assert that wherever the present form of the quantum theory is correct, our interpretation cannot lead to inconsistencies with relativity. In the domains where the present theory breaks down, there are several possible ways in which our interpretation could continue to treat the problem of covariance consistently. The attempt to maintain a consistent treatment of covariance in this problem might perhaps serve as an important heuristic principle in the search for new physical laws.

9. ON VON NEUMANN'S DEMONSTRATION THAT QUANTUM THEORY IS INCONSISTENT WITH HIDDEN VARIABLES

Von Neumann[6] has studied the following question: "If the present mathematical formulation of the quantum theory and its usual probability intepretation are assumed to lead to absolutely correct results for every experiment that can ever be done, can quantum-mechanical probabilities be understood in terms of any conceivable distribution over hidden parameters?" Von Neumann answers this question in the negative. His conclusions are subject, however, to the criticism that in his proof he has implicitly restricted himself to an excessively narrow class of hidden parameters and in this way has excluded from consideration precisely those types of hidden parameters which have been proposed in this paper.

To demonstrate the above statements, we summarize Von Neumann's proof briefly. This proof (which begins on p. 167 of his book), shows that the usual quantum-mechanical rules of calculating probabilities imply that there can be no "dispersionless states," i.e., states in which the values of all possible observables are simultaneously determined by physical parameters associated with the observed system. For example, if we consider two noncommuting observables, p and q, then Von Neumann shows that it would be inconsistent with the usual rules of calculating quantum-mechanical probabilities to assume that there were in the observed system a set of hidden parameters which simultaneously determined the results of measurements of position and momentum "observables." With this conclusion, we are in agreement. However, in our suggested new interpretation of the theory, the so-called "observables" are, as we have seen in Sec. 5, not properties belonging to the observed system alone, but instead potentialities whose precise development depends just as much on the observing apparatus as on the observed system. In fact, when we measure the momentum "observable," the final result is determined by hidden parameters in the momentum-measuring device as well as by hidden parameters in the observed electron. Similarly, when we measure the position "observable," the final result is determined in part by hidden parameters in the position-measuring device. Thus, the statistical distribution of "hidden" parameters to be used in calculating averages in a momentum measurement is different from the distribution to be used in calculating averages in a position measurement. Von Neumann's proof (see p. 171 in his book) that no single distribution of hidden parameters could be consistent with the results of the quantum theory is therefore irrelevant here, since in our interpretation of measurements of the type that can now be carried out, the distribution of hidden parameters varies in accordance with the different mutually exclusive experimental

[6] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Verlag. Julius Springer, Berlin, 1932).

arrangements of matter that must be used in making different kinds of measurements. In this point, we are in agreement with Bohr, who repeatedly stresses the fundamental role of the measuring apparatus as an inseparable part of the observed system. We differ from Bohr, however, in that we have proposed a method by which the role of the apparatus can be analyzed and described in principle in a precise way, whereas Bohr asserts that a precise conception of the details of the measurement process is as a matter of principle unattainable.

Finally, we wish to stress that the conclusions drawn thus far refer only to the measurement of the so-called "observables" carried out by the methods that are now available. If the quantum theory needs to be modified at small distances, then, as suggested in Sec. 6, precise measurements can in principle be made of the actual position and momentum of a particle. Here, it should be noted that Von Neumann's theorem is likewise irrelevant, this time because we are going beyond the assumption of the unlimited validity of the present general form of quantum theory, which plays an integral part in his proof.

10. SUMMARY AND CONCLUSIONS

The usual interpretation of the quantum theory implies that we must renounce the possibility of describing an individual system in terms of a single precisely defined conceptual model. We have, however, proposed an alternative interpretation which does not imply such a renunciation, but which instead leads us to regard a quantum-mechanical system as a synthesis of a precisely definable particle and a precisely definable ψ-field which exerts a force on this particle. An experimental choice between these two interpretations cannot be made in a domain in which the present mathematical formulation of the quantum theory is a good approximation; but such a choice is conceivable in domains, such as those associated with dimensions of the order of 10^{-13} cm, where the extrapolation of the present theory seems to break down and where our suggested new interpretation can lead to completely different kinds of predictions.

At present, our suggested new interpretation provides a consistent alternative to the usual assumption that no objective and precisely definable description of reality is possible at the quantum level of accuracy. For, in our description, the problem of objective reality at the quantum level is at least in principle not fundamentally different from that at the classical level, although new problems of measurement of the properties of an individual system appear, which can be solved only with the aid of an improvement in the theory, such as the possible modifications in the nuclear domain suggested in Sec. 6. In this connection, we wish to point out that what we can measure depends not only on the type of apparatus that is available, but also on the existing theory, which determines the kind of inference that can be used to connect the directly observable state of the apparatus with the state of the system of interest. In other words, our epistemology is determined to a large extent by the existing theory. It is therefore not wise to specify the possible forms of future theories in terms of purely epistomological limitations deduced from existing theories.

The development of the usual interpretation of the quantum theory seems to have been guided to a considerable extent by the principle of not postulating the possible existence of entities which cannot now be observed. This principle, which stems from a general philosophical point of view known during the nineteenth century as "positivism" or "empiricism" represents an extraphysical limitation on the possible kinds of theories that we shall choose to take into consideration.[7] The word "extraphysical" is used here advisedly, since we can in no way deduce, either from the experimental data of physics, or from its mathematical formulation, that it will necessarily remain forever impossible for us to observe entities whose existence cannot now be observed. Now, there is no reason why an extraphysical general principle is necessarily to be avoided, since such principles could conceivably serve as useful working hypotheses. The particular extraphysical principle described above cannot, however, be said to be a good working hypothesis. For the history of scientific research is full of examples in which it was very fruitful indeed to assume that certain objects or elements might be real, long before any procedures were known which would permit them to be observed directly. The atomic theory is just such an example. For the possibility of the actual existence of individual atoms was first postulated in order to explain various macrophysical results which could, however, also be understood directly in terms of macrophysical concepts without the need for assuming the existence of atoms. Certain nineteenth-century positivists (notably Mach) therefore insisted on purely philosophical grounds that it was incorrect to suppose that individual atoms actually existed, because they had never been observed as such. The atomic theory, they thought, should be regarded only as an interesting way of calculating various observable large-scale properties of matter. Nevertheless, evidence for the existence of individual atoms was ultimately discovered by people who took the atomic hypothesis seriously enough to suppose that individual atoms might actually exist, even though no one had yet observed them. We may have here, perhaps, a close analogy to the usual interpretation of the quantum theory, which avoids considering the possibility that the wave function of an individual system may represent objective reality, because we cannot observe it with the aid of existing experiments and theories.

[7] A leading nineteenth-century exponent of the positivist point of view was Mach. Modern positivists appear to have retreated from this extreme position, but its reflection still remains in the philosophical point of view implicitly adopted by a large number of modern theoretical physicists.

Finally, as an alternative to the positivist hypothesis of assigning reality only to that which we can now observe, we wish to prevent here another hypothesis, which we believe corresponds more closely to conclusions that can be drawn from general experience in actual scientific research. This hypothesis is based on the simple assumption that the world as a whole is objectively real and that, as far as we now know, it can correctly be regarded as having a precisely describable and analyzable structure of unlimited complexity. The pattern of this structure seems to be reflected completely but indirectly at every level, so that from experiments done at the level of size of human beings, it is very probably possible ultimately to draw inferences concerning the properties of the whole structure at all levels. We should never expect to obtain a complete theory of this structure, because there are almost certainly more elements in existence than we possibly can be aware of at any particular stage of scientific development. Any specified element, however, can in principle ultimately be discovered, but never all of them. Of course, we must avoid postulating a new element for each new phenomenon. But an equally serious mistake is to admit into the theory only those elements which can now be observed. For the purpose of a theory is not only to correlate the results of observations that we already know how to make, but also to suggest the need for new kinds of observations and to predict their results. In fact, the better a theory is able to suggest the need for new kinds of observations and to predict their results correctly, the more confidence we have that this theory is likely to be good representation of the actual properties of matter and not simply an empirical system especially chosen in such a way as to correlate a group of already known facts.

APPENDIX A. PHOTOELECTRIC AND COMPTON EFFECTS

In this appendix, we shall show how the electromagnetic field is to be described in our new interpretation, with the purpose of making possible a treatment of the photoelectric and Compton effects. For our purposes, it is adequate to restrict ourselves to a gauge in which $\operatorname{div}\mathbf{A}=0$, and to consider only the transverse part of the electromagnetic field, for in this gauge, the longitudinal part of the field can be expressed through Poisson's equation entirely in terms of the charge density. The Fourier analysis of the vector potential is then

$$\mathbf{A}(\mathbf{x})=(4\pi/V)^{\frac{1}{2}}\sum_{\mathbf{k},\mu}\epsilon_{\mathbf{k},\mu}q_{\mathbf{k},\mu}e^{i\mathbf{k}\cdot\mathbf{x}} \tag{A1}$$

with

$$q_{\mathbf{k},\mu}^*=q_{-\mathbf{k},\mu}.$$

The $q_{\mathbf{k},\mu}$ are coordinates of the electromagnetic field, associated with oscillations of wave number, $\mathbf{k}$, and polarization direction, μ, where $\boldsymbol{\epsilon}_{\mathbf{k},\mu}$ is a unit vector normal to $\mathbf{k}$ and μ runs over two indices, corresponding to a pair of orthogonal directions of polarization. V is the volume of the box, which is assumed to be very large.

We also introduce the momenta $\prod_{\mathbf{k},\mu}=\partial q_{\mathbf{k},\mu}^*/\partial t$, canonically conjugate[8] to the $q_{\mathbf{k},\mu}$. We have for the transverse part of the electric field

$$\mathfrak{E}(\mathbf{x})=-\frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x})}{\partial t}=-\left(\frac{4\pi}{Vc^2}\right)^{\frac{1}{2}}\sum_{\mathbf{k},\mu}\epsilon_{\mathbf{k},\mu}\prod_{\mathbf{k},\mu}^*e^{i\mathbf{k}\cdot\mathbf{x}} \tag{A2}$$

and for the magnetic field

$$\mathfrak{H}(\mathbf{x})=\nabla\times\mathbf{A}=-(4\pi/V)^{\frac{1}{2}}i\sum_{\mathbf{k},\mu}(\mathbf{k}\times\epsilon_{\mathbf{k},\mu})q_{\mathbf{k},\mu}e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{A3}$$

The Hamiltonian of the radiation field corresponds to a collection of independent harmonic oscillators, each with angular frequency, $\omega=kc$. This Hamiltonian is

$$H^{(R)}=\sum_{\mathbf{k},\mu}\left(\prod_{\mathbf{k},\mu}\prod_{\mathbf{k},\mu}^*+k^2c^2q_{\mathbf{k},\mu}q_{\mathbf{k},\mu}^*\right). \tag{A4}$$

Now, in our interpretation of the quantum theory, the quantity $q_{\mathbf{k},\mu}$ is assumed to refer to the actual value of the $\mathbf{k}$, μ Fourier component of the vector potential. As in the case of the electron, however, there is present an objectively real superfield that is a function of all the electromagnetic field coordinates $q_{\mathbf{k},\mu}$. Thus, we have

$$\Psi^{(R)}=\Psi^{(R)}(\cdots q_{\mathbf{k},\mu}\cdots). \tag{A5}$$

Writing $\Psi^{(R)}=R\exp(iS/\hbar)$, we obtain (in analogy with Paper I, Sec. 4)

$$\partial q_{\mathbf{k},\mu}/\partial t=\prod_{\mathbf{k},\mu}^*=\partial\mathfrak{S}/\partial q_{\mathbf{k},\mu}^*. \tag{A6}$$

The function $R(\cdots q_{\mathbf{k},\mu}\cdots)$ has two interpretations. First, it defines an additional quantum-mechanical term appearing in Maxwell's equations. To see the origin of this term, let us write the generalized Hamilton-Jacobi equation of the electromagnetic field, analogous to Paper I, Eq. (4),

$$\frac{\partial s}{\partial t}+\sum_{\mathbf{k},\mu}\frac{\partial s}{\partial q_{\mathbf{k},\mu}}\frac{\partial s}{\partial q_{\mathbf{k},\mu}^*}+\sum_{\mathbf{k},\mu}(kc)^2q_{\mathbf{k},\mu}q_{\mathbf{k},\mu}^*-\frac{\hbar^2}{2R}\sum_{\mathbf{k},\mu}\frac{\partial^2R(\cdots q_{\mathbf{k},\mu}\cdots)}{\partial q_{\mathbf{k},\mu}\partial q_{\mathbf{k},\mu}^*}=0. \tag{A7}$$

The equation of motion of $q_{\mathbf{k},\mu}$, derived from the Hamiltonian implied by Eq. (A7) becomes

$$\ddot{q}_{\mathbf{k},\mu}+k^2c^2q_{\mathbf{k},\mu}=\frac{\partial}{\partial q_{\mathbf{k},\mu}^*}\left(\frac{\hbar^2}{2R}\sum_{\mathbf{k}',\mu'}\frac{\partial^2R}{\partial q_{\mathbf{k}',\mu'}\partial q_{\mathbf{k}',\mu'}^*}\right). \tag{A8}$$

Since Maxwell's equations for empty space follow when the right-hand side is zero, we see that the "quantum-mechanical" terms can profoundly modify the behavior of the electromagnetic field. In fact, it is this modification which will contribute to the explana-

[8] See G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1948).

tion of the ability of an oscillator, $q_{\mathbf{k},\mu}$, to transfer large quantities of energy and momentum rapidly even when $q_{\mathbf{k},\mu}$ is very small, for when $q_{\mathbf{k},\mu}$ is small, the right-hand side of Eq. (A8) may become very large.

The second interpretation of R is that as in Paper I, Eq. (5), it defines a conserved probability density that each of the $q_{\mathbf{k},\mu}$ has a certain specified value. From this fact, we see that although large transfers of energy and momentum to a radiation oscillator can occur in a short time when R is small, the probability of such a process is (as was also shown in Paper I, Sec. 7) very small.

In the lowest state (when no quanta are present) every oscillator is in the ground state. The super wave fields is then

$$\Psi_0^{(R)}=\exp[-\textstyle\sum_{\mathbf{k},\mu}(kcq_{\mathbf{k},\mu}q_{\mathbf{k},\mu}^*+\frac{1}{2}ikct)]. \quad \text{(A9)}$$

If the $\mathbf{k}'$, μ' oscillator is excited to the nth quantum state, the super wave field is

$$\Psi^{(R)}=h_n(q_{\mathbf{k}',\mu'})e^{-ink'ct}\Psi_0^{(R)}, \quad \text{(A10)}$$

where h_n is the nth hermite polynomial. As shown in Paper I, Sec. 5, the stationary states of such a system correspond to a quantized energy equal to the same value, $E_n=(n+\frac{1}{2})\hbar kc$, obtained from the usual interpretation. In nonstationary states, however, Eqs. (A7) and (A8) imply that the energy of each oscillator may fluctuate violently, as was also true of nonstationary states of the hydrogen atom (see Paper I, Sec. 7).

A nonstationary state of particular interest in the photoelectric and Compton effects is a state corresponding to the presence of an electromagnetic wave packet containing a single quantum. The super wave field for such a state is

$$\Psi_P^{(R)}=\textstyle\sum_{\mathbf{k},\mu}f_\mu(\mathbf{k}-\mathbf{k}_0)q_{\mathbf{k},\mu}e^{-ikct}\Psi_0^{(R)}, \quad \text{(A11)}$$

where $f_\mu(\mathbf{k}-\mathbf{k}_0)$ is a function that is large only near $\mathbf{k}=\mathbf{k}_0$ and the first hermite polynomial is represented by $q_{\mathbf{k},\mu}$, to which it is proportional.

To prove that Eq. (A11) represents an electromagnetic wave packet, we can evaluate the difference

$$\langle\Delta W\rangle_{\text{Av}}=\langle W\rangle_{\text{Av}}-\langle W_0\rangle_{\text{Av}}, \quad \text{(A12)}$$

where $\langle W(\mathbf{x})\rangle_{\text{Av}}$ is the actual mean energy density present (averaged over the ensemble), and $\langle W_0(\mathbf{x})\rangle_{\text{Av}}$ is the mean energy that would be present even in the ground state, because of zero-point fluctuations. We have

$$\langle W(\mathbf{x})\rangle_{\text{Av}}=\int\int\cdots\int\Psi_P^{*(R)}(\cdots q_{\mathbf{k},\mu}\cdots)\times\frac{[\mathfrak{E}^2(\mathbf{x})+\mathfrak{H}^2(\mathbf{x})]}{8\pi}\Psi_P^{(R)}(\cdots q_{\mathbf{k},\mu}\cdots)\times(\cdots dq_{\mathbf{k},\mu}\cdots), \quad \text{(A13)}$$

$$\langle W_0(\mathbf{x})\rangle_{\text{Av}}=\int\int\cdots\int\Psi_0^{*(R)}(\cdots q_{\mathbf{k},\mu}\cdots)\times\frac{[\mathfrak{E}^2(\mathbf{x})+\mathfrak{H}^2(\mathbf{x})]}{8\pi}\Psi_0^{(R)}(\cdots q_{\mathbf{k},\mu}\cdots)\times(\cdots dq_{\mathbf{k},\mu}\cdots). \quad \text{(A14)}$$

Obtaining $\mathfrak{E}(\mathbf{x})$ from Eq. (A2), $\mathfrak{H}(\mathbf{x})$ from Eq. (A3), $\Psi_P^{(R)}$ from Eq. (A10), $\Psi_0^{(R)}$ from Eq. (A9), we readily show that

$$\langle\Delta W(\mathbf{x})\rangle_{\text{Av}}=\textstyle\sum_{\mathbf{k},\mu}\sum_{\mathbf{k}',\mu'}f_\mu(\mathbf{k}-\mathbf{k}_0)f_{\mu'}(\mathbf{k}'-\mathbf{k}_0)\times e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\epsilon_{\mathbf{k},\mu}\cdot\epsilon_{\mathbf{k}',\mu'}. \quad \text{(A15)}$$

This means that the wave packet implies an excess over zero-point energy that is localized within a region in which the packet function, $\mathbf{g}(\mathbf{x})$ is appreciable, where

$$\mathbf{g}(\mathbf{x})=\textstyle\sum_{\mathbf{k},\mu}f_\mu(\mathbf{k}-\mathbf{k}_0)e^{i\mathbf{k}\cdot\mathbf{x}}\epsilon_{\mathbf{k},\mu}. \quad \text{(A16)}$$

We are now ready to treat the photoelectric and Compton effects. The entire treatment is so similar to that of the Franck-Hertz experiment (Paper I, Sec. 7) that we need merely sketch it here. We begin by adding to the radiation Hamiltonian, $H^{(R)}$, the particle Hamiltonian,

$$H^{(P)}=(1/2m)[\mathbf{p}-(e/c)\mathbf{A}(\mathbf{x})]^2. \quad \text{(A17)}$$

(We restrict ourselves here to nonrelativistic treatment.) The photoelectric effect corresponds to the transition of a radiation oscillator from an excited state to the ground state, while the atomic electron is ejected, with an energy $E=h\nu-I$, where I is the ionization potential of the atom. The initial super wave field, corresponding to an incident packet containing only one quantum, plus an atom in the ground state is (see Eq. (A11))

$$\Psi_i=\psi_0(\mathbf{x})\exp(-iE_0t/\hbar)\Psi_0^{(R)}(\cdots q_{\mathbf{k},\mu}\cdots)\times\textstyle\sum_{\mathbf{k},\mu}f_\mu(\mathbf{k}-\mathbf{k}_0)q_{\mathbf{k},\mu}e^{-ikct}. \quad \text{(A18)}$$

By solving Schroedinger's equation for the combined system, we obtain an asymptotic wave field analogous to Paper I, Eq. (26), containing terms corresponding to the photoelectric effect. These terms, which must be added to Ψ_i, to yield the complete superfield, are (asymptotically)

$$\delta\Psi_a=\Psi_0^{(R)}(\cdots q_{\mathbf{k},\mu}\cdots)\sum_{\mathbf{k},\mu}f_\mu(\mathbf{k}-\mathbf{k}_0)\times\frac{\exp[i\mathbf{k}'\cdot\mathbf{r}-i\hbar(k'^2/2m)t]}{r}g_\mu(\theta,\phi,k'), \quad \text{(A19)}$$

where the energy of the outgoing electron is $E=\hbar^2k'^2/2m=\hbar kc+E_0$. The function $g_\mu(\theta,\phi,k')$ is the amplitude associated with the ψ-field of the outgoing electron. This quantity can be calculated from the matrix ele-

ment of the interaction term, $-(e/c)\mathbf{p}\cdot\mathbf{A}(\mathbf{x})$, by methods that are easily deducible from the usual perturbation theory.[6]

The outgoing electron packet has its center at $r=(\hbar k'/m)t$. Eventually, this packet will become completely separated from the initial electron wave function, $\psi_0(\mathbf{x})$. If the electron happens to enter the outgoing packet, the initial wave function can subsequently be ignored. The system then acts for all practical purposes as if its wave field were given by Eq. (A9), from which we conclude that the radiation field is in the ground state, while the electron has been liberated. It is readily shown that, as in the usual interpretation, the probability that the electron appears in the direction θ, ϕ can be calculated from $|g_\mu(\theta,\phi,k')|^2$ (see Paper I, Sec. 7).

To describe the Compton effect, we need only add to the super wave field the term corresponding to the appearance of an outgoing electromagnetic wave, as well as an outgoing electron. This part is asymptotically

$$\delta\Psi_b=\Psi_0^{(R)}(\cdots q_{\mathbf{k},\mu}\cdots)\sum_{\mathbf{k}',\mu'}f_\mu(\mathbf{k}-\mathbf{k}_0)\times c_{\mathbf{k}',\mu'}{}^{\mathbf{k},\mu}q_{k',\mu'}g_{\mathbf{k}',\mu'}{}^{\mathbf{k},\mu}(\theta,\phi)\frac{e^{ik''r}}{r}\times\exp\left(-ik'ct-\frac{i\hbar k''^2t}{2m}\right),\quad\text{(A20)}$$

where

$$(\hbar^2k''^2/2m)+\hbar k'c=\hbar kc+E_0.$$

The quantity, $c_{\mathbf{k}',\mu'}{}^{\mathbf{k},\mu}$ is proportioned to the matrix element for a transition in which the $\mathbf{k}$, μ-radiation oscillator falls from the first excited state, to the ground state, while the $\mathbf{k}'$, μ'-oscillator rises from the ground state to the first excited state. This matrix element is determined mainly by the term $(e^2/8mc^2)A^2(\mathbf{x})$ in the hamiltonian.

It is easily seen that the outgoing electron packet eventually becomes completely separated both from the initial wave field, $\Psi_i(\mathbf{x},\cdots q_{\mathbf{k},\mu}\cdots)$, and from the packet for the photoelectric effect, $\delta\Psi_a$ [defined in Eq. (A19)]. If the electron should happen to enter this packet, then the others can be ignored, and the system acts for all practical purposes like an outgoing electron, plus an independent outgoing light quantum. The reader will readily verify that the probability that the light quantum $\mathbf{k}'$, μ' appears along with an electron with angles θ, ϕ is precisely the same as in the usual interpretation.

APPENDIX B. A DISCUSSION OF INTERPRETATIONS OF THE QUANTUM THEORY PROPOSED BY DE BROGLIE AND ROSEN

After this article had been prepared, the author's attention was called to two papers in which an interpretation of the quantum theory similar to that suggested here was proposed, first by L. de Broglie,[9] and later by N. Rosen.[10] In both of these papers, it was suggested that if one writes $\psi=R\exp(is/\hbar)$, then one can regard R^2 as a probability density of particles having a velocity, $\mathbf{v}=\nabla s/m$. De Broglie regarded the ψ-field as an agent "guiding" the particle, and therefore referred to ψ as a "pilot wave." Both of these authors came to the conclusion that this interpretation could not consistently be carried through in those cases in which the field contained a linear combination of stationary state wave functions. As we shall see in this appendix, however, the difficulties encountered by the above authors could have been overcome by them, if only they had carried their ideas to a logical conclusion.

De Broglie's suggestions met strong objections on the part of Pauli,[11] in connection with the problem of inelastic scattering of a particle by a rigid rotator. Since this problem is conceptually equivalent to that of inelastic scattering of a particle by a hydrogen atom, which we have already treated in Paper I, Sec. 7, we shall discuss the objections raised by Pauli in terms of the latter example.

Now, according to Pauli's argument, the initial wave function in the scattering problem should be $\Psi=\exp(i\mathbf{p}_0\cdot\mathbf{y}/\hbar)\psi_0(\mathbf{x})$. This corresponds to a stationary state for the combined system, in which the particle momentum is $\mathbf{p}_0$, while the hydrogen atom is in its ground state, with a wave function, $\psi_0(\mathbf{x})$. After interaction between the incident particle and the hydrogen atom, the combined wave function can be represented as

$$\Psi=\sum_n f_n(\mathbf{y})\psi_n(\mathbf{x}),\quad\text{(B1)}$$

where $\psi_n(\mathbf{x})$ is the wave function for the nth excited state of the hydrogen atom, and $f_n(\mathbf{y})$ is the associated expansion coefficient. It is easily shown[12] that asymptotically, $f_n(\mathbf{y})$ takes the form of an outgoing wave, $f_n(\mathbf{y})\sim g_n(\theta,\phi)e^{iknr}/r$, where $(\hbar kn)^2/2m=[(\hbar k_0)^2/2m]+E_n-E_0$. Now, if we write $\psi=R\exp(iS/\hbar)$, we find that the particle momenta, $p_x=\nabla_xS(\mathbf{x},\mathbf{y})$ and $\mathbf{p}_y=\nabla_yS(\mathbf{x},\mathbf{y})$, fluctuate violently in a way that depends strongly on the position of each particle. Thus, neither atom nor the outgoing particle ever seem to approach a stationary energy. On the other hand, we know from experiment that both the atom and the outgoing particle do eventually obtain definite (but presumably unpredictable) energy values. Pauli therefore concluded that the interpretation proposed by de Broglie was untenable. De Broglie seems to have agreed with the conclusion, since he subsequently gave up his suggested interpretation.[9]

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[9] L. de Broglie, *An Introduction to the Study of Wave Mechanics* (E. P. Dutton and Company, Inc., New York, 1930), see Chapters 6, 9, and 10.

[10] N. Rosen, J. Elisha Mitchel Sci. Soc. **61**, Nos. 1 and 2 (August, 1945).

[11] *Reports on the 1927 Solvay Congress* (Gauthiers-Villars et Cie., Paris, 1928), see p. 280.

[12] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1933).

Our answer to Pauli's objection is already contained in Paper I, Sec. 7, as well as in Sec. 2 of this paper. For as is well known, the use of an incident plane wave of infinite extent is an excessive abstraction, not realizable in practice. Actually, both the incident and outgoing parts of the ψ-field will always take the form of bounded packets. Moreover, as shown in Paper I, Sec. 7, all packets corresponding to different values of n will ultimately obtain classically describable separations. The outgoing particle must enter one of these packets, and it will remain with that particular packet thereafter, leaving the hydrogen atom in a definite but correlated stationary state. Thus, Pauli's objection is seen to be based on the use of the excessively abstract model of an infinite plane wave.

Although the above constitutes a complete answer to Pauli's specific objections to our suggested interpretation, we wish here to amplify our discussion somewhat, in order to anticipate certain additional objections that might be made along similar lines. For at this point, one might argue that even though the wave packet is bounded, it can nevertheless in principle be made arbitrarily large in extent by means of a suitable adjustment of initial conditions. Our interpretation predicts that in the region in which incident and outgoing ψ-waves overlap, the momentum of each particle will fluctuate violently, as a result of corresponding fluctuations in the "quantum-mechanical" potential produced by the ψ-field. The question arises, however, as to whether such fluctuations can really be in accord with experimental fact, especially since in principle they could occur when the particles were separated by distances much greater than that over which the "classical" interaction potential, $V(\mathbf{x}, \mathbf{y})$, was appreciable.

To show that these fluctuations are not in disagreement with any experimental facts now available, we first point out that even in the usual interpretation the energy of each particle cannot correctly be regarded as definite under the conditions which are assumed here, namely, that the incident and outgoing wave packets overlap. For as long as interference between two stationary state wave function is possible, the system acts as if it, in some sense, covered both states simultaneously.[13] In such a situation, the usual interpretation implies that a precisely defined value for the energy of either particle is meaningless. From such a wave function, one can predict only the probability that if the energy is measured, a definite value will be obtained. On the other hand, the very experimental conditions needed for measuring the energy play a key role in making a definite value of the energy possible because the effect of the measuring apparatus is to destroy interference between parts of the wave function corresponding to different values of the energy.[14]

[13] Reference 2, Chapter 16, Sec. 25.
[14] Reference 2, Chapter 6, Secs. 3 to 8; Chapter 22, Secs. 8 to 10.

In our interpretation, the overlap of incident and outgoing wave packets signifies not that the precise value of the energy of either particle can be given no meaning, but rather that this value fluctuates violently in an, in practice, unpredictable and uncontrollable way. When the energy of either particle is measured, however, then our interpretation predicts, in agreement with the usual interpretation, that the energy of each particle will become definite and constant, as a result of the effects of the energy-measuring apparatus on the observed system. To show how this happens, let us suppose that the energy of the hydrogen atom is measured by means of an interaction in which the "classical" potential, V, is a function only of the variables associated with the electron in the hydrogen atom and with the apparatus, but is not a function of variables associated with the outgoing particle. Let z be the coordinate of the measuring apparatus. Then as shown in Sec. 2, interaction with an apparatus that measures the energy of the hydrogen atom will transform the Ψ-function (B1), into

$$\Psi=\sum_n f_n(\mathbf{y})\psi_n(\mathbf{x})g_0(z-aEnt/\hbar^2). \qquad \text{(B2)}$$

Now, we have seen that if the product at is large enough to make a distinct measurement possible, packets corresponding to different values of n will ultimately obtain classically describable separations in z space. The apparatus variable, z, must enter one of these packets; and, thereafter, all other packets can for practical purposes be ignored. The hydrogen atom is then left in a state having a definite and constant energy, while the outgoing particle has a correspondingly definite but correlated constant value for its energy. Thus, we find that as with the usual interpretation, our interpretation predicts that whenever we measure the energy of either particle by methods that are now available, a definite and constant value will always be obtained. Nevertheless, under conditions in which incident and outgoing wave packets overlap, and in which neither particle interacts with an energy-measuring device, our interpretation states unambiguously that real fluctuations in the energy of each particle will occur. These fluctuations are moreover, at least in principle, observable (for example, by methods discussed in Sec. 6). Meanwhile, under conditions in which we are limited by present methods of observation, our interpretation leads to predictions that are precisely the same as those obtained from the usual interpretation, so that no experiments supporting the usual interpretation can possibly contradict our interpretation.

In his book,[9] de Broglie raises objections to his own suggested interpretation of the quantum theory, which are very similar to those raised by Pauli. It is therefore not necessary to answer de Broglie's objections in detail here, since the answer is essentially the same as that which has been given to Pauli. We wish, however,

to add one point. De Broglie assumes that not only electrons, but also light quanta, are associated with particles. A consistent application of the interpretation suggested here requires, however, as shown in Appendix A, that light quanta be described as electromagnetic wave packets. The only precisely definable quantities in such a packet are the Fourier components, $q_{\mathbf{k},\mu}$, of the vector potential and the corresponding canonically conjugate momenta, $\Pi_{\mathbf{k},\mu}$. Such packets have many particle-like properties, including the ability to transfer rapidly a full quantum of energy at great distances. Nevertheless, it would not be consistent to assume the existence of a "photon" particle, associated with each light quantum.

We shall now discuss Rosen's paper briefly.[10] Rosen gave up his suggested interpretation of the quantum theory, because of difficulties arising in connection with the interpretation of standing waves. In the case of the stationary states of a free particle in a box, which we have already discussed in Sec. 8, our interpretation leads to the conclusion that the particle is standing still. Rosen did not wish to accept this conclusion, because it seemed to disagree with the statement of the usual interpretation that in such a state the electron is moving with equal probability that the motion is in either direction. To answer Rosen's objections, we need merely point out again that the usual interpretation can give no meaning to the motion of particles in a stationary state; at best, it can only predict the probability that a given result will be obtained, if the velocity is measured. As we saw in Sec. 8, however, our interpretation leads to precisely the same predictions as are obtained from the usual interpretation, for any process which could actually provide us with a measurement of the velocity of the electron. One must remember, however, that the value of the momentum "observable" as it is now "measured" is not necessarily equal to the particle momentum existing before interaction with the measuring apparatus took place.

We conclude that the objections raised by Pauli, de Broglie, and Rosen, to interpretations of the quantum theory similar to that suggested here, can all be answered by carrying every aspect of our suggested interpretation to its logical conclusion.

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Neutrons from the Disintegration of the Separated Isotopes of Silicon by Deuterons*

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The neutron spectra from deuterons on silicon have been studied by the method of recoil protons and photographic plates. Thick isotopic targets of the three separated isotopes of silicon were irradiated by deuterons of energy 1.4 Mev, supplied by the Bartol Van de Graaff statitron, observations being carried out at angles of zero and ninety degrees with the incident deuterons. Q-values, from which energy levels in the residual nuclei of phosphorous may be calculated, are as follows:

Reaction	Q-values (Mev)
$Si^{28}(d,n)P^{29}$	0.29
$Si^{29}(d,n)P^{30}$	3.27, 2.52, 1.81, 1.27
$Si^{30}(d,n)P^{31}$	4.92, 4.59, 3.73, 2.70, 1.51

The estimated probable error in the Q-values is 40 kev.

THERE are three stable isotopes of silicon, Si^{28}, Si^{29}, and Si^{30}, having respectively relative abundances of 92.28 percent, 4.67 percent, and 3.05 percent. When these elements are irradiated by deuterons, neutrons are emitted in the following three reactions.

$$(1)\ Si^{28}+d\rightarrow P^{29}+n^1+Q_1,$$
$$(2)\ Si^{29}+d\rightarrow P^{30}+n^1+Q_2,$$
$$(3)\ Si^{30}+d\rightarrow P^{31}+n^1+Q_3.$$

Naturally occurring silicon has been previously bombarded with deuterons to observe the neutron spectra.[1] However, because of the element of ambiguity introduced by the presence of the mixture of isotopes, the data are difficult of interpretation. To reinvestigate the above three reactions, quantities of the separated isotopes, in the form of silicon dioxide, were obtained from the Y-12 plant, Carbide and Carbon Chemicals Division, Union Carbide and Carbon Corporation, Oak Ridge, Tennessee. The mass analyses of the various targets are shown in Table I.

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‡ The contribution of D. M. Van Patter to this article consists of the preparation of the Appendix.

[1] R. A. Peck, Jr., Phys. Rev. 73, 947 (1948).