A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I

DAVID BOHM*
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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The usual interpretation of the quantum theory is self-consistent, but it involves an assumption that cannot be tested experimentally, *viz.*, that the most complete possible specification of an individual system is in terms of a wave function that determines only probable results of actual measurement processes. The only way of investigating the truth of this assumption is by trying to find some other interpretation of the quantum theory in terms of at present "hidden" variables, which in principle determine the precise behavior of an individual system, but which are in practice averaged over in measurements of the types that can now be carried out. In this paper and in a subsequent paper, an interpretation of the quantum theory in terms of just such "hidden" variables is suggested. It is shown that as long as the mathematical theory retains its present general form, this suggested interpretation leads to precisely the same results for all physical processes as does the usual interpretation. Nevertheless, the suggested interpretation provides a broader conceptual framework than the usual interpretation, because it makes possible a precise and continuous description of all processes, even at the quantum level. This broader conceptual framework allows more general mathematical formulations of the theory than those allowed by the usual interpretation. Now, the usual mathematical formulation seems to lead to insoluble difficulties when it is extrapolated into the domain of distances of the order of 10^{-13} cm or less. It is therefore entirely possible that the interpretation suggested here may be needed for the resolution of these difficulties. In any case, the mere possibility of such an interpretation proves that it is not necessary for us to give up a precise, rational, and objective description of individual systems at a quantum level of accuracy.

1. INTRODUCTION

THE usual interpretation of the quantum theory is based on an assumption having very far-reaching implications, *viz.*, that the physical state of an individual system is completely specified by a wave function that determines only the probabilities of actual results that can be obtained in a statistical ensemble of similar experiments. This assumption has been the object of severe criticisms, notably on the part of Einstein, who has always believed that, even at the quantum level, there must exist precisely definable elements or dynamical variables determining (as in classical physics) the actual behavior of each individual system, and not merely its probable behavior. Since these elements or variables are not now included in the quantum theory and have not yet been detected experimentally, Einstein has always regarded the present form of the quantum theory as incomplete, although he admits its internal consistency.[1–5]

Most physicists have felt that objections such as those raised by Einstein are not relevant, first, because the present form of the quantum theory with its usual probability interpretation is in excellent agreement with an extremely wide range of experiments, at least in the domain of distances[6] larger than 10^{-13} cm, and, secondly, because no consistent alternative interpretations have as yet been suggested. The purpose of this paper (and of a subsequent paper hereafter denoted by II) is, however, to suggest just such an alternative interpretation. In contrast to the usual interpretation, this alternative interpretation permits us to conceive of each individual system as being in a precisely definable state, whose changes with time are determined by definite laws, analogous to (but not identical with) the classical equations of motion. Quantum-mechanical probabilities are regarded (like their counterparts in classical statistical mechanics) as only a practical necessity and not as a manifestation of an inherent lack of complete determination in the properties of matter at the quantum level. As long as the present general form of Schroedinger's equation is retained, the physical results obtained with our suggested alternative interpretation are precisely the same as those obtained with the usual interpretation. We shall see, however, that our alternative interpretation permits modifications of the mathematical formulation which could not even be described in terms of the usual interpretation. Moreover, the modifications can quite easily be formulated in such a way that their effects are insignificant in the atomic domain, where the present quantum theory is in such good agreement with experiment, but of crucial importance in the domain of dimensions of the order of 10^{-13} cm, where, as we have seen, the present theory is totally inadequate. It is thus entirely possible that some of the modifications describable in terms of our suggested alternative interpretation, but

* Now at Universidade de São Paulo, Faculdade de Filosofia, Ciencias, e Letras, São Paulo, Brasil.

[1] Einstein, Podolsky, and Rosen, Phys. Rev. 47, 777 (1933).

[2] D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., New York, 1951), see p. 611.

[3] N. Bohr, Phys. Rev. 48, 696 (1935).

[4] W. Furry, Phys. Rev. 49, 393, 476 (1936).

[5] Paul Arthur Schilp, editor, *Albert Einstein, Philosopher-Scientist* (Library of Living Philosophers, Evanston, Illinois, 1949). This book contains a thorough summary of the entire controversy.

[6] At distances of the order of 10^{-13} cm or smaller and for times of the order of this distance divided by the velocity of light or smaller, present theories become so inadequate that it is generally believed that they are probably not applicable, except perhaps in a very crude sense. Thus, it is generally expected that in connection with phenomena associated with this so-called "fundamental length," a totally new theory will probably be needed. It is hoped that this theory could not only deal precisely with such processes as meson production and scattering of elementary particles, but that it would also systematically predict the masses, charges, spins, etc., of the large number of so-called "elementary" particles that have already been found, as well as those of new particles which might be found in the future.

not in terms of the usual interpretation, may be needed for a more thorough understanding of phenomena associated with very small distances. We shall not, however, actually develop such modifications in any detail in these papers.

After this article was completed, the author's attention was called to similar proposals for an alternative interpretation of the quantum theory made by de Broglie[7] in 1926, but later given up by him partly as a result of certain criticisms made by Pauli[8] and partly because of additional objections raised by de Broglie[7] himself.† As we shall show in Appendix B of Paper II, however, all of the objections of de Broglie and Pauli could have been met if only de Broglie had carried his ideas to their logical conclusion. The essential new step in doing this is to apply our interpretation in the theory of the measurement process itself as well as in the description of the observed system. Such a development of the theory of measurements is given in Paper II,[9] where it will be shown in detail that our interpretation leads to precisely the same results for all experiments as are obtained with the usual interpretation. The foundation for doing this is laid in Paper I, where we develop the basis of our interpretation, contrast it with the usual interpretation, and apply it to a few simple examples, in order to illustrate the principles involved.

2. THE USUAL PHYSICAL INTERPRETATION OF THE QUANTUM THEORY

The usual physical interpretation of the quantum theory centers around the uncertainty principle. Now, the uncertainty principle can be derived in two different ways. First, we may start with the assumption already criticized by Einstein,[1] namely, that a wave function that determines only probabilities of actual experimental results nevertheless provides the most complete possible specification of the so-called "quantum state" of an individual system. With the aid of this assumption and with the aid of the de Broglie relation, $\mathbf{p}=\hbar\mathbf{k}$, where $\mathbf{k}$ is the wave number associated with a particular fourier component of the wave function, the uncertainty principle is readily deduced.[10] From this derivation, we are led to interpret the uncertainty principle as an inherent and irreducible limitation on the precision with which it is correct for us even to conceive of momentum and position as simultaneously defined quantities. For if, as is done in the usual interpretation of the quantum theory, the wave intensity is assumed to determine only the probability of a given position, and if the $\mathbf{k}$th Fourier component of the wave function is assumed to determine only the probability of a corresponding momentum, $\mathbf{p}=\hbar\mathbf{k}$, then it becomes a contradiction in terms to ask for a state in which momentum and position are simultaneously and precisely defined.

A second possible derivation of the uncertainty principle is based on a theoretical analysis of the processes with the aid of which physically significant quantities such as momentum and position can be measured. In such an analysis, one finds that because the measuring apparatus interacts with the observed system by means of indivisible quanta, there will always be an irreducible disturbance of some observed property of the system. If the precise effects of this disturbance could be predicted or controlled, then one could correct for these effects, and thus one could still in principle obtain simultaneous measurements of momentum and position, having unlimited precision. But if one could do this, then the uncertainty principle would be violated. The uncertainty principle is, as we have seen, however, a necessary consequence of the assumption that the wave function and its probability interpretation provide the most complete possible specification of the state of an individual system. In order to avoid the possibility of a contradiction with this assumption, Bohr[3,5,10,11] and others have suggested an additional assumption, namely, that the process of transfer of a single quantum from observed system to measuring apparatus is inherently unpredictable, uncontrollable, and not subject to a detailed rational analysis or description. With the aid of this assumption, one can show[10] that the same uncertainty principle that is deduced from the wave function and its probability interpretation is also obtained as an inherent and unavoidable limitation on the precision of all possible measurements. Thus, one is able to obtain a set of assumptions, which permit a self-consistent formulation of the usual interpretation of the quantum theory.

The above point of view has been given its most consistent and systematic expression by Bohr,[3,5,10] in terms of the "principle of complementarity." In formulating this principle, Bohr suggests that at the atomic level we must renounce our hitherto successful practice of conceiving of an individual system as a unified and precisely definable whole, all of whose aspects are, in a manner of speaking, simultaneously and

[7] L. de Broglie, *An Introduction to the Study of Wave Mechanics* (E. P. Dutton and Company, Inc., New York, 1930), see Chapters 6, 9, and 10. See also Compt. rend. **183**, 447 (1926); **184**, 273 (1927); **185**, 380 (1927).

[8] *Reports on the Solvay Congress* (Gauthiers-Villars et Cie., Paris, 1928), see p. 280.

† *Note added in proof.*—Madelung has also proposed a similar interpretation of the quantum theory, but like de Broglie he did not carry this interpretation to a logical conclusion. See E. Madelung, Z. f. Physik **40**, 332 (1926), also G. Temple, *Introduction to Quantum Theory* (London, 1931).

[9] In Paper II, Sec. 9, we also discuss von Neumann's proof [see J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Verlag, Julius Springer, Berlin, 1932)] that quantum theory cannot be understood in terms of a statistical distribution of "hidden" causal parameters. We shall show that his conclusions do not apply to our interpretation, because he implicitly assumes that the hidden parameters must be associated only with the observed system, whereas, as will become evident in these papers, our interpretation requires that the hidden parameters shall also be associated with the measuring apparatus.

[10] See reference 2, Chapter 5.

[11] N. Bohr, *Atomic Theory and the Description of Nature* (Cambridge University Press, London, 1934).

unambiguously accessible to our conceptual gaze. Such a system of concepts, which is sometimes called a "model," need not be restricted to pictures, but may also include, for example, mathematical concepts, as long as these are supposed to be in a precise (i.e., one-to-one) correspondence with the objects that are being described. The principle of complementarity requires us, however, to renounce even mathematical models. Thus, in Bohr's point of view, the wave function is in no sense a conceptual model of an individual system, since it is not in a precise (one-to-one) correspondence with the behavior of this system, but only in a statistical correspondence.

In place of a precisely defined conceptual model, the principle of complementarity states that we are restricted to complementarity pairs of inherently imprecisely defined concepts, such as position and momentum, particle and wave, etc. The maximum degree of precision of definition of either member of such a pair is reciprocally related to that of the opposite member. This need for an inherent lack of complete precision can be understood in two ways. First, it can be regarded as a consequence of the fact that the experimental apparatus needed for a precise measurement of one member of a complementary pair of variables must always be such as to preclude the possibility of a simultaneous and precise measurement of the other member. Secondly, the assumption that an individual system is completely specified by the wave function and its probability interpretation implies a corresponding unavoidable lack of precision in the very conceptual structure, with the aid of which we can think about and describe the behavior of the system.

It is only at the classical level that we can correctly neglect the inherent lack of precision in all of our conceptual models; for here, the incomplete determination of physical properties implied by the uncertainty principle produces effects that are too small to be of practical significance. Our ability to describe classical systems in terms of precisely definable models is, however, an integral part of the usual interpretation of the theory. For without such models, we would have no way to describe, or even to think of, the result of an observation, which is of course always finally carried out at a classical level of accuracy. If the relationships of a given set of classically describable phenomena depend significantly on the essentially quantum-mechanical properties of matter, however, then the principle of complementarity states that no single model is possible which could provide a precise and rational analysis of the connections between these phenomena. In such a case, we are not supposed, for example, to attempt to describe in detail how future phenomena arise out of past phenomena. Instead, we should simply accept without further analysis the fact that future phenomena do in fact somehow manage to be produced, in a way that is, however, necessarily beyond the possibility of a detailed description. The only aim of a mathematical theory is then to predict the statistical relations, if any, connecting these phenomena.

3. CRITICISM OF THE USUAL INTERPRETATION OF THE QUANTUM THEORY

The usual interpretation of the quantum theory can be criticized on many grounds.[5] In this paper, however, we shall stress only the fact that it requires us to give up the possibility of even conceiving precisely what might determine the behavior of an individual system at the quantum level, without providing adequate proof that such a renunciation is necessary.[9] The usual interpretation is admittedly consistent; but the mere demonstration of such consistency does not exclude the possibility of other equally consistent interpretations, which would involve additional elements or parameters permitting a detailed causal and continuous description of all processes, and not requiring us to forego the possibility of conceiving the quantum level in precise terms. From the point of view of the usual interpretation, these additional elements or parameters could be called "hidden" variables. As a matter of fact, whenever we have previously had recourse to statistical theories, we have always ultimately found that the laws governing the individual members of a statistical ensemble could be expressed in terms of just such hidden variables. For example, from the point of view of macroscopic physics, the coordinates and momenta of individual atoms are hidden variables, which in a large scale system manifest themselves only as statistical averages. Perhaps then, our present quantum-mechanical averages are similarly a manifestation of hidden variables, which have not, however, yet been detected directly.

Now it may be asked why these hidden variables should have so long remained undetected. To answer this question, it is helpful to consider as an analogy the early forms of the atomic theory, in which the existence of atoms was postulated in order to explain certain large-scale effects, such as the laws of chemical combination, the gas laws, etc. On the other hand, these same effects could also be described directly in terms of existing macrophysical concepts (such as pressure, volume, temperature, mass, etc.); and a correct description in these terms did not require any reference to atoms. Ultimately, however, effects were found which contradicted the predictions obtained by extrapolating certain purely macrophysical theories to the domain of the very small, and which could be understood correctly in terms of the assumption that matter is composed of atoms. Similarly, we suggest that if there are hidden variables underlying the present quantum theory, it is quite likely that in the atomic domain, they will lead to effects that can also be described adequately in the terms of the usual quantum-mechanical concepts; while in a domain associated with much smaller dimensions, such as the level associated with the "fundamental length" of the order of 10^{-13} cm, the hidden variables

may lead to completely new effects not consistent with the extrapolation of the present quantum theory down to this level.

If, as is certainly entirely possible, these hidden variables are actually needed for a correct description at small distances, we could easily be kept on the wrong track for a long time by restricting ourselves to the usual interpretation of the quantum theory, which excludes such hidden variables as a matter of principle. It is therefore very important for us to investigate our reasons for supposing that the usual physical interpretation is likely to be the correct one. To this end, we shall begin by repeating the two mutually consistent assumptions on which the usual interpretation is based (see Sec. 2):

(1) The wave function with its probability interpretation determines the most complete possible specification of the state of an individual system.

(2) The process of transfer of a single quantum from observed system to measuring apparatus is inherently unpredictable, uncontrollable, and unanalyzable.

Let us now inquire into the question of whether there are any experiments that could conceivably provide a test for these assumptions. It is often stated in connection with this problem that the mathematical apparatus of the quantum theory and its physical interpretation form a consistent whole and that this combined system of mathematical apparatus and physical interpretation is tested adequately by the extremely wide range of experiments that are in agreement with predictions obtained by using this system. If assumptions (1) and (2) implied a unique mathematical formulation, then such a conclusion would be valid, because experimental predictions could then be found which, if contradicted, would clearly indicate that these assumptions were wrong. Although assumptions (1) and (2) do limit the possible forms of the mathematical theory, they do not limit these forms sufficiently to make possible a unique set of predictions that could in principle permit such an experimental test. Thus, one can contemplate practically arbitrary changes in the Hamiltonian operator, including, for example, the postulation of an unlimited range of new kinds of meson fields each having almost any conceivable rest mass, charge, spin, magnetic moment, etc. And if such postulates should prove to be inadequate, it is conceivable that we may have to introduce nonlocal operators, nonlinear fields, S-matrices, etc. This means that when the theory is found to be inadequate (as now happens, for example, at distances of the order of 10^{-13} cm), it is always possible, and, in fact, usually quite natural, to assume that the theory can be made to agree with experiment by some as yet unknown change in the mathematical formulation alone, not requiring any fundamental changes in the physical interpretation. This means that as long as we accept the usual physical interpretation of the quantum theory, we cannot be led by any conceivable experiment to give up this interpretation, even if it should happen to be wrong. The usual physical interpretation therefore presents us with a considerable danger of falling into a trap, consisting of a self-closing chain of circular hypotheses, which are in principle unverifiable if true. The only way of avoiding the possibility of such a trap is to study the consequences of postulates that contradict assumptions (1) and (2) at the outset. Thus, we could, for example, postulate that the precise outcome of each individual measurement process is in principle determined by some at present "hidden" elements or variables; and we could then try to find experiments that depended in a unique and reproducible way on the assumed state of these hidden elements or variables. If such predictions are verified, we should then obtain experimental evidence favoring the hypothesis that hidden variables exist. If they are not verified, however, the correctness of the usual interpretation of the quantum theory is not necessarily proved, since it may be necessary instead to alter the specific character of the theory that is supposed to describe the behavior of the assumed hidden variables.

We conclude then that a choice of the present interpretation of the quantum theory involves a real physical limitation on the kinds of theories that we wish to take into consideration. From the arguments given here, however, it would seem that there are no secure experimental or theoretical grounds on which we can base such a choice because this choice follows from hypotheses that cannot conceivably be subjected to an experimental test and because we now have an alternative interpretation.

4. NEW PHYSICAL INTERPRETATION OF SCHROEDINGER'S EQUATION

We shall now give a general description of our suggested physical interpretation of the present mathematical formulation of the quantum theory. We shall carry out a more detailed description in subsequent sections of this paper.

We begin with the one-particle Schroedinger equation, and shall later generalize to an arbitrary number of particles. This wave equation is

$$i\hbar\partial\psi/\partial t=-(\hbar^2/2m)\nabla^2\psi+V(\mathbf{x})\psi. \tag{1}$$

Now ψ is a complex function, which can be expressed as

$$\psi=R\exp(iS/\hbar), \tag{2}$$

where R and S are real. We readily verify that the equations for R and S are

$$\frac{\partial R}{\partial t}=-\frac{1}{2m}[R\nabla^2S+2\nabla R\cdot\nabla S], \tag{3}$$

$$\frac{\partial S}{\partial t}=-\left[\frac{(\nabla S)^2}{2m}+V(\mathbf{x})-\frac{\hbar^2}{2m}\frac{\nabla^2R}{R}\right]. \tag{4}$$

It is convenient to write $P(\mathbf{x})=R^2(\mathbf{x})$, or $R=P^{\frac{1}{2}}$ where $P(\mathbf{x})$ is the probability density. We then obtain

$$\frac{\partial P}{\partial t}+\nabla\cdot\left(P\frac{\nabla S}{m}\right)=0, \tag{5}$$

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2m}+V(\mathbf{x})-\frac{\hbar^2}{4m}\left[\frac{\nabla^2 P}{P}-\frac{1}{2}\frac{(\nabla P)^2}{P^2}\right]=0. \tag{6}$$

Now, in the classical limit ($\hbar\to 0$) the above equations are subject to a very simple interpretation. The function $S(\mathbf{x})$ is a solution of the Hamilton-Jacobi equation. If we consider an ensemble of particle trajectories which are solutions of the equations of motion, then it is a well-known theorem of mechanics that if all of these trajectories are normal to any given surface of constant S, then they are normal to all surfaces of constant S, and $\nabla S(\mathbf{x})/m$ will be equal to the velocity vector, $\mathbf{v}(\mathbf{x})$, for any particle passing the point $\mathbf{x}$. Equation (5) can therefore be re-expressed as

$$\partial P/\partial t+\nabla\cdot(P\mathbf{v})=0. \tag{7}$$

This equation indicates that it is consistent to regard $P(\mathbf{x})$ as the probability density for particles in our ensemble. For in that case, we can regard $P\mathbf{v}$ as the mean current of particles in this ensemble, and Eq. (7) then simply expresses the conservation of probability.

Let us now see to what extent this interpretation can be given a meaning even when $\hbar\neq 0$. To do this, let us assume that each particle is acted on, not only by a "classical" potential, $V(\mathbf{x})$ but also by a "quantum-mechanical" potential,

$$U(\mathbf{x})=\frac{-\hbar^2}{4m}\left[\frac{\nabla^2 P}{P}-\frac{1}{2}\frac{(\nabla P)^2}{P^2}\right]=\frac{-\hbar^2}{2m}\frac{\nabla^2 R}{R}. \tag{8}$$

Then Eq. (6) can still be regarded as the Hamilton-Jacobi equation for our ensemble of particles, $\nabla S(\mathbf{x})/m$ can still be regarded as the particle velocity, and Eq. (5) can still be regarded as describing conservation of probability in our ensemble. Thus, it would seem that we have here the nucleus of an alternative interpretation for Schroedinger's equation.

The first step in developing this interpretation in a more explicit way is to associate with each electron a particle having precisely definable and continuously varying values of position and momentum. The solution of the modified Hamilton-Jacobi equation (4) defines an ensemble of possible trajectories for this particle, which can be obtained from the Hamilton-Jacobi function, $S(\mathbf{x})$, by integrating the velocity, $\mathbf{v}(\mathbf{x})=\nabla S(\mathbf{x})/m$. The equation for S implies, however, that the particles moves under the action of a force which is not entirely derivable from the classical potential, $V(\mathbf{x})$, but which also obtains a contribution from the "quantum-mechanical" potential, $U(\mathbf{x})=(-\hbar^2/2m)\times\nabla^2 R/R$. The function, $R(\mathbf{x})$, is not completely arbitrary, but is partially determined in terms of $S(\mathbf{x})$ by the differential Eq. (3). Thus R and S can be said to codetermine each other. The most convenient way of obtaining R and S is, in fact, usually to solve Eq. (1) for the Schroedinger wave function, ψ, and then to use the relations,

$$\psi=U+iW=R[\cos(S/\hbar)+i\sin(S/\hbar)],$$
$$R^2=U^2+V^2;\quad S=\hbar\tan^{-1}(W/U).$$

Since the force on a particle now depends on a function of the absolute value, $R(\mathbf{x})$, of the wave function, $\psi(\mathbf{x})$, evaluated at the actual location of the particle, we have effectively been led to regard the wave function of an individual electron as a mathematical representation of an objectively real field. This field exerts a force on the particle in a way that is analogous to, but not identical with, the way in which an electromagnetic field exerts a force on a charge, and a meson field exerts a force on a nucleon. In the last analysis, there is, of course, no reason why a particle should not be acted on by a ψ-field, as well as by an electromagnetic field, a gravitational field, a set of meson fields, and perhaps by still other fields that have not yet been discovered.

The analogy with the electromagnetic (and other) field goes quite far. For just as the electromagnetic field obeys Maxwell's equations, the ψ-field obeys Schroedinger's equation. In both cases, a complete specification of the fields at a given instant over every point in space determines the values of the fields for all times. In both cases, once we know the field functions, we can calculate force on a particle, so that, if we also know the initial position and momentum of the particle, we can calculate its entire trajectory.

In this connection, it is worth while to recall that the use of the Hamilton-Jacobi equation in solving for the motion of a particle is only a matter of convenience and that, in principle, we can always solve directly by using Newton's laws of motion and the correct boundary conditions. The equation of motion of a particle acted on by the classical potential, $V(\mathbf{x})$, and the "quantum-mechanical" potential, Eq. (8), is

$$md^2\mathbf{x}/dt^2=-\nabla\{V(\mathbf{x})-(\hbar^2/2m)\nabla^2 R/R\}. \tag{8a}$$

It is in connection with the boundary conditions appearing in the equations of motion that we find the only fundamental difference between the ψ-field and other fields, such as the electromagnetic field. For in order to obtain results that are equivalent to those of the usual interpretation of the quantum theory, we are required to restrict the value of the initial particle momentum to $\mathbf{p}=\nabla S(\mathbf{x})$. From the application of Hamilton-Jacobi theory to Eq. (6), it follows that this restriction is consistent, in the sense that if it holds initially, it will hold for all time. Our suggested new interpretation of the quantum theory implies, however, that this restriction is not inherent in the conceptual structure. We shall see in Sec. 9, for example, that it is

quite consistent in our interpretation to contemplate modifications in the theory, which permit an arbitrary relation between $\mathbf{p}$ and $\nabla S(\mathbf{x})$. The law of force on the particle can, however, be so chosen that in the atomic domain, $\mathbf{p}$ turns out to be very nearly equal to $\nabla S(\mathbf{x})/m$, while in processes involving very small distances, these two quantities may be very different. In this way, we can improve the analogy between the ψ-field and the electromagnetic field (as well as between quantum mechanics and classical mechanics).

Another important difference between the ψ-field and the electromagnetic field is that, whereas Schroedinger's equation is homogeneous in ψ, Maxwell's equations are inhomogeneous in the electric and magnetic fields. Since inhomogeneities are needed to give rise to radiation, this means that our present equations imply that the ψ-field is not radiated or absorbed, but simply changes its form while its integrated intensity remains constant. This restriction to a homogeneous equation is, however, like the restriction to a homogeneous equation is, however, like the restriction to $\mathbf{p}=\nabla S(\mathbf{x})$, not inherent in the conceptual structure of our new interpretation. Thus, in Sec. 9, we shall show that one can consistently postulate inhomogeneities in the equation governing ψ, which produce important effects only at very small distances, and negligible effects in the atomic domain. If such inhomogeneities are actually present, then the ψ-field will be subject to being emitted and absorbed, but only in connection with processes associated with very small distances. Once the ψ-field has been emitted, however, it will in all atomic processes simply obey Schroedinger's equation as a very good approximation. Nevertheless, at very small distances, the value of the ψ-field would, as in the case of the electromagnetic field, depend to some extent on the actual location of the particle.

Let us now consider the meaning of the assumption of a statistical ensemble of particles with a probability density equal to $P(\mathbf{x})=R^2(\mathbf{x})=|\psi(\mathbf{x})|^2$. From Eq. (5), it follows that this assumption is consistent, provided that ψ satisfies Schroedinger's equation, and $\mathbf{v}=\nabla S(\mathbf{x})/m$. This probability density is numerically equal to the probability density of particles obtained in the usual interpretation. In the usual interpretation, however, the need for a probability description is regarded as inherent in the very structure of matter (see Sec. 2), whereas in our interpretation, it arises, as we shall see in Paper II, because from one measurement to the next, we cannot in practice predict or control the precise location of a particle, as a result of corresponding unpredictable and uncontrollable disturbances introduced by the measuring apparatus. Thus, in our interpretation, the use of a statistical ensemble is (as in the case of classical statistical mechanics) only a practical necessity, and not a reflection of an inherent limitation on the precision with which it is correct for us to conceive of the variables defining the state of the system. Moreover, it is clear that if in connection with very small distances we are ultimately required to give up the special assumptions that ψ satisfies Schroedinger's equation and that $\mathbf{v}=\nabla S(\mathbf{x})/m$, then $|\psi|^2$ will cease to satisfy a conservation equation and will therefore also cease to be able to represent the probability density of particles. Nevertheless, there would still be a true probability density of particles which is conserved. Thus, it would become possible in principle to find experiments in which $|\psi|^2$ could be distinguished from the probability density, and therefore to prove that the usual interpretation, which gives $|\psi|^2$ only a probability interpretation must be inadequate. Moreover, we shall see in Paper II that with the aid of such modifications in the theory, we could in principle measure the particle positions and momenta precisely, and thus violate the uncertainty principle. As long as we restrict ourselves to conditions in which Schroedinger's equation is satisfied, and in which $\mathbf{v}=\nabla S(\mathbf{x})/m$, however, the uncertainty principle will remain an effective practical limitation on the possible precision of measurements. This means that at present, the particle positions and momenta should be regarded as "hidden" variables, since as we shall see in Paper II, we are not now able to obtain experiments that localize them to a region smaller than that in which the intensity of the ψ-field is appreciable. Thus, we cannot yet find clear-cut experimental proof that the assumption of these variables is necessary, although it is entirely possible that, in the domain of very small distances, new modifications in the theory may have to be introduced, which would permit a proof of the existence of the definite particle position and momentum to be obtained.

We conclude that our suggested interpretation of the quantum theory provides a much broader conceptual framework than that provided by the usual interpretation, for all of the results of the usual interpretation are obtained from our interpretation if we make the following three special assumptions which are mutually consistent:

(1) That the ψ-field satisfies Schroedinger's equation.

(2) That the particle momentum is restricted to $\mathbf{p}=\nabla S(\mathbf{x})$.

(3) That we do not predict or control the precise location of the particle, but have, in practice, a statistical ensemble with probability density $P(\mathbf{x})=|\psi(\mathbf{x})|^2$. The use of statistics is, however, not inherent in the conceptual structure, but merely a consequence of our ignorance of the precise initial conditions of the particle.

As we shall see in Sec. 9, it is entirely possible that a better theory of phenomena involving distances of the order of 10^{-13} cm or less would require us to go beyond the limitations of these special assumptions. Our principal purpose in this paper (and in Paper II) is to show, however, that if one makes these special assumptions, our interpretation leads in all possible experiments to the same predictions as are obtained from the usual interpretation.[9]

It is now easy to understand why the adoption of the

usual interpretation of the quantum theory would tend to lead us away from the direction of our suggested alternative interpretation. For in a theory involving hidden variables, one would normally expect that the behavior of an individual system should not depend on the statistical ensemble of which it is a member, because this ensemble refers to a series of similar but disconnected experiments carried out under equivalent initial conditions. In our interpretation, however, the "quantum-mechanical" potential, $U(\mathbf{x})$, acting on an individual particle depends on a wave intensity, $P(\mathbf{x})$, that is also numerically equal to a probability density in our ensemble. In the terminology of the usual interpretation of the quantum theory, in which one tacitly assumes that the wave function has only one interpretation; namely, in terms of a probability, our suggested new interpretation would look like a mysterious dependence of the individual on the statistical ensemble of which it is a member. In our interpretation, such a dependence is perfectly rational, because the wave function can consistently be interpreted both as a force and as a probability density.[12]

It is instructive to carry our analogy between the Schroedinger field and other kinds of fields a bit further. To do this, we can derive the wave Eqs. (5) and (6) from a Hamiltonian functional. We begin by writing down the expression for the mean energy as it is expressed in the usual quantum theory:

$$\bar{H}=\int\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})\right)\psi d\mathbf{x}$$
$$=\int\left\{\frac{\hbar^2}{2m}|\nabla\psi|^2+V(\mathbf{x})|\psi|^2\right\}d\mathbf{x}.$$

Writing $\psi=P^{\frac{1}{2}}\exp(iS/\hbar)$, we obtain

$$\bar{H}=\int P(\mathbf{x})\left\{\frac{(\nabla S)^2}{2m}+V(\mathbf{x})+\frac{\hbar^2}{8m}\frac{(\nabla P)^2}{P^2}\right\}d\mathbf{x}. \quad (9)$$

We shall now reinterpret $P(\mathbf{x})$ as a field coordinate, defined at each point, $\mathbf{x}$, and we shall tentatively assume that $S(\mathbf{x})$ is the momentum, canonically conjugate to $P(\mathbf{x})$. That such an assumption is appropriate can be verified by finding the Hamiltonian equations of motion for $P(\mathbf{x})$ and $S(\mathbf{x})$, under the assumption that the Hamiltonian functional is equal to $\bar{H}$ (See Eq. (9)). These equations of motion are

$$\dot{P}=\frac{\delta\bar{H}}{\delta S}=-\frac{1}{m}\nabla\cdot(P\nabla S),$$
$$\dot{S}=-\frac{\delta\bar{H}}{\delta P}=-\left[\frac{(\nabla S)^2}{2m}+V(\mathbf{x})-\frac{\hbar^2}{4m}\left(\frac{\nabla^2P}{P}-\frac{1}{2}\frac{(\nabla P)^2}{P^2}\right)\right].$$

[12] This consistency is guaranteed by the conservation Eq. (7). The questions of why an arbitrary statistical ensemble tends to decay into an ensemble with a probability density equal to $\psi^*\psi$ will be discussed in Paper II, Sec. 7.

These are, however, the same as the correct wave Eqs. (5) and (6).

We can now show that the mean particle energy averaged over our ensemble is equal to the usual quantum mechanical mean value of the Hamiltonian, $\bar{H}$. To do this, we note that according to Eqs. (3) and (6), the energy of a particle is

$$E(\mathbf{x})=-\frac{\partial S(\mathbf{x})}{\partial t}=\left[\frac{(\nabla S)^2}{2m}+V(\mathbf{x})-\frac{\hbar^2}{2m}\frac{\nabla^2R}{R}\right]. \quad (10)$$

The mean particle energy is found by averaging $E(\mathbf{x})$ with the weighting function, $P(\mathbf{x})$. We obtain

$$\langle E\rangle_{\substack{\text{ensemble}\\\text{average}}}=\int P(\mathbf{x})E(\mathbf{x})d\mathbf{x}$$
$$=\int P(\mathbf{x})\left[\frac{(\nabla S)^2}{2m}+V(\mathbf{x})\right]d\mathbf{x}-\frac{\hbar^2}{2m}\int R\nabla^2Rd\mathbf{x}.$$

A little integration by parts yields

$$\langle E\rangle_{\substack{\text{ensemble}\\\text{average}}}=\int P(\mathbf{x})\left[\frac{(\nabla S)^2}{2m}+V(\mathbf{x})+\frac{\hbar^2}{8m}\frac{(\nabla P)^2}{P^2}\right]d\mathbf{x}=\bar{H}. \quad (11)$$

5. THE STATIONARY STATE

We shall now show how the problem of stationary states is to be treated in our interpretation of the quantum theory.

The following seem to be reasonable requirements in our interpretation for a stationary state:

(1) The particle energy should be a constant of the motion.

(2) The quantum-mechanical potential should be independent of time.

(3) The probability density in our statistical ensemble should be independent of time.

It is easily verified that these requirements can be satisfied with the assumption that

$$\psi(\mathbf{x},t)=\psi_0(\mathbf{x})\exp(-iEt/\hbar)$$
$$=R_0(\mathbf{x})\exp[i(\Phi(\mathbf{x})-Et)/\hbar]. \quad (12)$$

From the above, we obtain $S=\Phi(\mathbf{x})-Et$. According to the generalized Hamilton-Jacobi Eq. (4), the particle energy is given by

$$\partial S/\partial t=-E.$$

Thus, we verify that the particle energy is a constant of the motion. Moreover, since $P=R^2=|\psi|^2$, it follows that P (and R) are independent of time. This means that both the probability density in our ensemble and the quantum-mechanical potential are also time independent.

The reader will readily verify that no other form of solution of Schroedinger's equation will satisfy all three of our criteria for a stationary state.

Since ψ is now being regarded as a mathematical representation of an objectively real force field, it follows that (like the electromagnetic field) it should be everywhere finite, continuous, and single valued. These requirements will guarantee in all cases that occur in practice that the allowed values of the energy in a stationary state, and the corresponding eigenfunctions are the same as are obtained from the usual interpretation of the theory.

In order to show in more detail what a stationary state means in our interpretation, we shall now consider three examples of stationary states.

Case 1: "s" State

The first case that we shall consider is an "s" state. In an "s" state, the wave function is

$$\psi=f(r)\exp[i(\alpha-Et)/\hbar], \tag{13}$$

where α is an arbitrary constant and r is the radius taken from the center of the atom. We conclude that the Hamilton-Jacobi function is

$$S=\alpha-Et.$$

The particle velocity is

$$\mathbf{v}=\nabla S=0.$$

The particle is therefore simply standing still, wherever it may happen to be. How can it do this? The absence of motion is possible because the applied force, $-\nabla V(\mathbf{x})$, is balanced by the "quantum-mechanical" force, $(\hbar^2/2m)\nabla(\nabla^2R/R)$, produced by the Schroedinger ψ-field acting on its own particle. There is, however, a statistical ensemble of possible positions of the particle, with a probability density, $P(\mathbf{x})=(f(r))^2$.

Case 2: State with Nonzero Angular Momentum

In a typical state of nonzero angular momentum, we have

$$\psi=f_n{}^l(r)P_l{}^m(\cos\theta)\exp[i(\beta-Et+\hbar m\phi)/\hbar], \tag{14}$$

where θ and ϕ are the colatitude and azimuthal polar angles, respectively, $P_l{}^m$ is the associated Legendre polynomial, and β is a constant. The Hamilton-Jacobi function is $S=\beta-Et+\hbar m\phi$. From this result it follows that the z component of the angular momentum is equal to $\hbar m$. To prove this, we write

$$L_z=xp_y-yp_x=x\partial S/\partial y-y\partial S/\partial x=\partial S/\partial\phi=\hbar m. \tag{15}$$

Thus, we obtain a statistical ensemble of trajectories which can have different forms, but all have the same "quantized" value of the z component of the angular momentum.

Case 3: A Scattering Problem

Let us now consider a scattering problem. Because it is comparatively easy to analyze, we shall discuss a hypothetical experiment, in which an electron is incident in the z direction with an initial momentum, p_0, on a system consisting of two slits.[13] After the electron passes through the slit system, its position is measured and recorded, for example, on a photographic plate.

Now, in the usual interpretation of the quantum theory, the electron is described by a wave function. The incident part of the wave function is $\psi_0\sim\exp(ip_0z/\hbar)$; but when the wave passes through the slit system, it is modified by interference and diffraction effects, so that it will develop a characteristic intensity pattern by the time it reaches the position measuring instrument. The probability that the electron will be detected between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ is $|\psi(\mathbf{x})|^2d\mathbf{x}$. If the experiment is repeated many times under equivalent initial conditions, one eventually obtains a pattern of hits on the photographic plate that is very reminiscent of the interference patterns of optics.

In the usual interpretation of the quantum theory, the origin of this interference pattern is very difficult to understand. For there may be certain points where the wave function is zero when both slits are open, but not zero when only one slit is open. How can the opening of a second slit prevent the electron from reaching certain points that it could reach if this slit were closed? If the electron acted completely like a classical particle, this phenomenon could not be explained at all. Clearly, then the wave aspects of the electron must have something to do with the production of the interference pattern. Yet, the electron cannot be identical with its associated wave, because the latter spreads out over a wide region. On the other hand, when the electron's position is measured, it always appears at the detector as if it were a localized particle.

The usual interpretation of the quantum theory not only makes no attempt to provide a single precisely defined conceptual model for the production of the phenomena described above, but it asserts that no such model is even conceivable. Instead of a single precisely defined conceptual model, it provides, as pointed out in Sec. 2, a pair of complementary models, *viz.*, particle and wave, each of which can be made more precise only under conditions which necessitate a reciprocal decrease in the degree of precision of the other. Thus, while the electron goes through the slit system, its position is said to be inherently ambiguous, so that if we wish to obtain an interference pattern, it is meaningless to ask through which slit an individual electron actually passed. Within the domain of space within which the position of the electron has no meaning we can use the wave model and thus describe the subsequent production of interference. If, however, we

[13] This experiment is discussed in some detail in reference 2, Chapter 6, Sec. 2.

tried to define the position of the electron as it passed the slit system more accurately by means of a measurement, the resulting disturbance of its motion produced by the measuring apparatus would destroy the interference pattern. Thus, conditions would be created in which the particle model becomes more precisely defined at the expense of a corresponding decrease in the degree of definition of the wave model. When the position of the electron is measured at the photographic plate, a similar sharpening of the degree of definition of the particle model occurs at the expense of that of the wave model.

In our interpretation of the quantum theory, this experiment is described causally and continuously in terms of a single precisely definable conceptual model. As we have already shown, we must use the same wave function as is used in the usual interpretation; but instead we regard it as a mathematical representation of an objectively real field that determines part of the force acting on the particle. The initial momentum of the particle is obtained from the incident wave function, $\exp(ip_0z/\hbar)$, as $p=\partial s/\partial z=p_0$. We do not in practice, however, control the initial location of the particle, so that although it goes through a definite slit, we cannot predict which slit this will be. The particle is at all times acted on by the "quantum-mechanical" potential, $U=(-\hbar^2/2m)\nabla^2R/R$. While the particle is incident, this potential vanishes because R is then a constant; but after it passes through the slit system, the particle encounters a quantum-mechanical potential that changes rapidly with position. The subsequent motion of the particle may therefore become quite complicated. Nevertheless, the probability that a particle shall enter a given region, $d\mathbf{x}$, is as in the usual interpretation, equal to $|\psi(\mathbf{x})|^2d\mathbf{x}$. We therefore deduce that the particle can never reach a point where the wave function vanishes. The reason is that the "quantum-mechanical" potential, U, becomes infinite when R becomes zero. If the approach to infinity happens to be through positive values of U, there will be an infinite force repelling the particle away from the origin. If the approach is through negative values of U, the particle will go through this point with infinite speed, and thus spend no time there. In either case, we obtain a simple and precisely definable conceptual model explaining why particles can never be found at points where the wave function vanishes.

If one of the slits is closed, the "quantum-mechanical" potential is correspondingly altered, because the ψ-field is changed, and the particle may then be able to reach certain points which it was unable to reach when both slits were open. The slit is therefore able to affect the motion of the particle only indirectly, through its effect on the Schroedinger ψ-field. Moreover, as we shall see in Paper II, if the position of the electron is measured while it is passing through the slit system, the measuring apparatus will, as in the usual interpretation, create a disturbance that destroys the interference pattern. In our interpretation, however, the necessity for this destruction is not inherent in the conceptual structure; and as we shall see, the destruction of the interference pattern could in principle be avoided by means of other ways of making measurements, ways which are conceivable but not now actually possible.

6. THE MANY-BODY PROBLEM

We shall now extend our interpretation of the quantum theory to the problem of many bodies. We begin with the Schroedinger equation for two particles. (For simplicity, we assume that they have equal masses, but the extension of our treatment to arbitrary masses will be obvious.)

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}(\nabla_1^2\psi+\nabla_2^2\psi)+V(\mathbf{x}_1,\mathbf{x}_2)\psi.$$

Writing $\psi=R(\mathbf{x}_1,\mathbf{x}_2)\exp[iS(\mathbf{x}_1,\mathbf{x}_2)/\hbar]$ and $R^2=P$, we obtain

$$\frac{\partial P}{\partial t}+\frac{1}{m}[\nabla_1\cdot P\nabla_1S+\nabla_2\cdot P\nabla_2S]=0, \tag{16}$$

$$\frac{\partial S}{\partial t}+\frac{(\nabla_1S)^2+(\nabla_2S)^2}{2m}+V(\mathbf{x}_1,\mathbf{x}_2)-\frac{\hbar^2}{2mR}[\nabla_1^2R+\nabla_2^2R]=0. \tag{17}$$

The above equations are simply a six-dimensional generalization of the similar three-dimensional Eqs. (5) and (6) associated with the one-body problem. In the two-body problem, the system is described therefore by a six-dimensional Schroedinger wave and by a six-dimensional trajectory, specifying the actual location of each of the two particles. The velocity of this trajectory has components, ∇_1S/m and ∇_2S/m, respectively, in each of the three-dimensional surfaces associated with a given particle. $P(\mathbf{x}_1,\mathbf{x}_2)$ then has a dual interpretation. First, it defines a "quantum-mechanical" potential, acting on each particle

$$U(\mathbf{x}_1,\mathbf{x}_2)=-(\hbar^2/2mR)[\nabla_1^2R+\nabla_2^2R].$$

This potential introduces an additional effective interaction between particles over and above that due to the classically inferrable potential $V(\mathbf{x})$. Secondly, the function $P(\mathbf{x}_1,\mathbf{x}_2)$ can consistently be regarded as the probability density of representative points $(\mathbf{x}_1,\mathbf{x}_2)$ in our six-dimensional ensemble.

The extension to an arbitrary number of particles is straightforward, and we shall quote only the results here. We introduce the wave function, $\psi=R(\mathbf{x}_1,\mathbf{x}_2,\cdots\mathbf{x}_n)\exp[iS(\mathbf{x}_1,\mathbf{x}_2\cdots\mathbf{x}_n)/\hbar]$ and define a $3n$-dimensional trajectory, where n is the number of particles, which describes the behavior of every particle in the system. The velocity of the ith particle is $\mathbf{v}_i=\nabla_iS(\mathbf{x}_1,\mathbf{x}_2\cdots\mathbf{x}_n)/m$. The function $P(\mathbf{x}_1,\mathbf{x}_2\cdots\mathbf{x}_n)=R^2$ has two

interpretations. First, it defines a "quantum-mechanical" potential

$$U(\mathbf{x}_1, \mathbf{x}_2 \cdots \mathbf{x}_n) = -\frac{\hbar^2}{2mR} \sum_{s=1}^{n} \nabla_s^2 R(\mathbf{x}_1, \mathbf{x}_2 \cdots \mathbf{x}_n). \quad (18)$$

Secondly, $P(\mathbf{x}_1, \mathbf{x}_2 \cdots \mathbf{x}_n)$ is equal to the density of representative points $(\mathbf{x}_1, \mathbf{x}_2 \cdots \mathbf{x}_n)$ in our $3n$-dimensional ensemble.

We see here that the "effective potential," $U(\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_n)$, acting on a particle is equivalent to that produced by a "many-body" force, since the force between any two particles may depend significantly on the location of every other particle in the system. An example of the effects of such a force is given by the exclusion principle. Thus, if the wave function is antisymmetric, we deduce that the "quantum-mechanical" forces will be such as to prevent two particles from ever reaching the same point in space, for in this case, we must have $P=0$.

7. TRANSITIONS BETWEEN STATIONARY STATES—THE FRANCK-HERTZ EXPERIMENT

Our interpretation of the quantum theory describes all processes as basically causal and continuous. How then can it lead to a correct description of processes such as the Franck-Hertz experiment, the photoelectric effect, and the Compton effect, which seem to call most strikingly for an interpretation in terms of discontinuous and incompletely determined transfers of energy and momentum? In this section, we shall answer this question by applying our suggested interpretation of the quantum theory in the analysis of the Franck-Hertz experiment. Here, we shall see that the apparently discontinuous nature of the process of transfer of energy from the bombarding particle to the atomic electron is brought about by the "quantum-mechanical" potential, $U=(-\hbar^2/2m)\nabla^2 R/R$, which does not necessarily become small when the wave intensity becomes small. Thus, even if the force of interaction between two particles is very weak, so that a correspondingly small disturbance of the Schroedinger wave function is produced by the interaction of these particles, this disturbance is capable of bringing about very large transfers of energy and momentum between the particles in a very short time. This means that if we view only the end results, this process presents the aspect of being discontinuous. Moreover, we shall see that the precise value of the energy transfer is in principle determined by the initial position of each particle and by the initial form of the wave function. Since we cannot in practice predict or control the initial particle positions with complete precision, we are also unable to predict or control the final outcome of such an experiment, and can, in practice, predict only the probability of a given outcome. Because the probability that the particles will enter a region with coordinates, $\mathbf{x}_1$, $\mathbf{x}_2$, is proportional to $R^2(\mathbf{x}_1, \mathbf{x}_2)$, we conclude that although a Schroedinger wave of low intensity can bring about large transfers of energy, such a process is (as in the usual interpretation) highly improbable.

In Appendix A of Paper II, we shall see that similar possibilities arise in connection with the interaction of the electromagnetic field with charged matter, so that electromagnetic waves can very rapidly transfer a full quantum of energy (and momentum) to an electron, even after they have spread out and fallen to a very low intensity. In this way, we shall explain the photoelectric effect and the Compton effect. Thus, we are able in our interpretation to understand by means of a causal and continuous model just those properties of matter and light which seem most convincingly to require the assumption of discontinuity and incomplete determinism.

Before we discuss the process of interaction between two particles, we shall find it convenient to analyze the problem of an isolated single particle that happens to be in a nonstationary state. Because the field function ψ is a solution of Schroedinger's equation, we can linearly suppose stationary-state solutions of this equation and in this way obtain new solutions. As an illustration, let us consider a superposition of two solutions

$$\psi = C_1\psi_1(\mathbf{x})\exp(-iE_1t/\hbar) + C_2\psi_2(\mathbf{x})\exp(-iE_2t/\hbar),$$

where C_1, C_2, ψ_1, and ψ_2 are real. Thus we write $\psi_1 = R_1$, $\psi_2 = R_2$, and

$$\psi = \exp[-i(E_1+E_2)t/2\hbar]\{C_1R_1\exp[-i(E_1-E_2)t/2\hbar] + C_2R_2\exp[i(E_1-E_2)t/2\hbar]\}.$$

Writing $\psi = R\exp(iS/\hbar)$, we obtain

$$R^2 = C_1^2R_1^2(\mathbf{x}) + C_2^2R_2^2(\mathbf{x}) + 2C_1C_2R_1(\mathbf{x})R_2(\mathbf{x})\cos[(E_1-E_2)t/2\hbar], \quad (19)$$

$$\tan\left\{\frac{S+(E_1-E_2)t/2}{\hbar}\right\} = \frac{C_2R_2(\mathbf{x})-C_1R_1(\mathbf{x})}{C_2R_2(\mathbf{x})+C_1R_1(\mathbf{x})}\tan\left\{\frac{(E_1-E_2)t}{2\hbar}\right\}. \quad (20)$$

We see immediately that the particle experiences a "quantum-mechanical" potential, $U(\mathbf{x}) = (-\hbar/2m)\nabla^2 R/R$, which fluctuates with angular frequency, $w=(E_1-E_2)/\hbar$, and that the energy of this particle, $E=-\partial S/\partial t$, and its momentum $\mathbf{p}=\nabla S$, fluctuate with the same angular frequency. If the particle happens to enter a region of space where R is small, these fluctuations can become quite violent. We see then that, in general, the orbit of a particle in a nonstationary state is very irregular and complicated, resembling Brownian motion more closely than it resembles the smooth track of a planet around the sun.

If the system is isolated, these fluctuations will continue forever. The result is quite reasonable, since as is well known, a system can make a transition from one stationary state to another only if it can exchange en-

ergy with some other system. In order to treat the problem of transition between stationary states, we must therefore introduce another system capable of exchanging energy with the system of interest. In this section, we shall discuss the Franck-Hertz experiment, in which this other system consists of a bombarding particle. For the sake of illustration, let us suppose that we have hydrogen atoms of energy E_0 and wave function, $\psi_0(\mathbf{x})$, which are bombarded by particles that can be scattered inelastically, leaving the atom with energy E_n and wave function, $\psi_n(\mathbf{x})$.

We begin by writing down the initial wave function, $\Psi_i(\mathbf{x}, \mathbf{y}, t)$. The incident particle, whose coordinates are represented by $\mathbf{y}$ must be associated with a wave packet, which can be written as

$$f_0(\mathbf{y}, t)=\int e^{i\mathbf{k}\cdot\mathbf{y}} f(\mathbf{k}-\mathbf{k}_0)\exp(-i\hbar k^2 t/2m)d\mathbf{k}. \quad (21)$$

The center of this packet occurs where the phase has an extremum as a function of $\mathbf{k}$, or where $\mathbf{y}=\hbar\mathbf{k}_0 t/m$.

Now, as in the usual interpretation, we begin by writing the incident wave function for the combined system as a product

$$\Psi_i=\psi_0(\mathbf{x})\exp(-iE_0 t/\hbar) f_0(\mathbf{y}, t). \quad (22)$$

Let us now see how this wave function is to be understood in our interpretation of the theory. As pointed out in Sec. 6, the wave function is to be regarded as a mathematical representation of a six-dimensional but objectively real field, capable of producing forces that act on the particles. We also assume a six-dimensional representative point, described by the coordinates of the two particles, $\mathbf{x}$ and $\mathbf{y}$. We shall now see that when the combined wave function takes the form (22) involving a product of a function of $\mathbf{x}$ and a function of $\mathbf{y}$, the six-dimensional system can correctly be regarded as being made up of two independent three-dimentional subsystems. To prove this, we write

$$\psi_0(\mathbf{x})=R_0(\mathbf{x})\exp[iS_0(\mathbf{x})/\hbar] \quad \text{and}$$
$$f_0(\mathbf{y}, t)=M_0(\mathbf{y}, t)\exp[iN_0(\mathbf{y}, t)/\hbar].$$

We then obtain for the particle velocities

$$d\mathbf{x}/dt=(1/m)\nabla S_0(\mathbf{x}); \quad d\mathbf{y}/dt=(1/m)\nabla N_0(y, t), \quad (23)$$

and for the "quantum-mechanical" potential

$$U=-\frac{\hbar^2\{(\nabla_x^2+\nabla_y^2)R(\mathbf{x}, \mathbf{y})\}}{2mR(\mathbf{x}, \mathbf{y})}$$
$$=\frac{-\hbar^2}{2m}\left\{\frac{\nabla^2 R_0(\mathbf{x})}{R_0(\mathbf{x})}+\frac{\nabla^2 M_0(\mathbf{y}, t)}{M_0(\mathbf{y}, t)}\right\}. \quad (24)$$

Thus, the particle velocities are independent and the "quantum-mechanical" potential reduces to a sum of terms, one involving only $\mathbf{x}$ and the other involving only $\mathbf{y}$. This means that the particles move independently. Moreover, the probability density, $P=R_0^2(\mathbf{x})\times M_0^2(\mathbf{y}, t)$, is a product of a function of $\mathbf{x}$ and a function of $\mathbf{y}$, indicating that the distribution in $\mathbf{x}$ is statistically independent of that in $\mathbf{y}$. We conclude, then, that whenever the wave function can be expressed as a product of two factors, each involving only the coordinates of a single system, then the two systems are completely independent of each other.

As soon as the wave packet in $\mathbf{y}$ space reaches the neighborhood of the atom, the two systems begin to interact. If we solve Schroedinger's equation for the combined system, we obtain a wave function that can be expressed in terms of the following series:

$$\Psi=\Psi_i+\sum_n\psi_n(\mathbf{x})\exp(-iE_n t/\hbar) f_n(\mathbf{y}, t), \quad (25)$$

where the $f_n(\mathbf{y}, t)$ are the expansion coefficients of the complete set of functions, $\psi_n(\mathbf{x})$. The asymptotic form of the wave function is[14]

$$\Psi=\Psi_i(\mathbf{x}, \mathbf{y})+\sum_n\psi_n(\mathbf{x})\exp\left(-\frac{iE_n t}{\hbar}\right)\int f(\mathbf{k}-\mathbf{k}_0)$$
$$\times\frac{\exp[ik_n\cdot\mathbf{r}-(\hbar k_n^2/2n)t]}{r}g_n(\theta, \phi, \mathbf{k})d\mathbf{k}, \quad (26)$$

where

$$\hbar^2 k_n^2/2m=(\hbar^2 k_0^2/2m)+E_0-E_n$$
(conservation of energy). (27)

The additional terms in the above equation represent outgoing wave packets, in which the particle speed, $\hbar k_n/m$, is correlated with the wave function, $\psi_n(\mathbf{x})$, representing the state in which the hydrogen atom is left. The center of the nth packet occurs at

$$r_n=(\hbar k_n/m)t. \quad (28)$$

It is clear that because the speed depends on the hydrogen atom quantum number, n, every one of these packets will eventually be separated by distances which are so large that this separation is classically describable.

When the wave function takes the form (25), the two particles system must be described as a single six-dimensional system and not as a sum of two independent three-dimensional subsystems, for at this time, if we try to express the wave function as $\psi(\mathbf{x}, \mathbf{y})=R(\mathbf{x}, \mathbf{y})\times\exp[iS(\mathbf{x}, \mathbf{y})/\hbar]$, we find that the resulting expressions for R and S depend on $\mathbf{x}$ and $\mathbf{y}$ in a very complicated way. The particle momenta, $\mathbf{p}_1=\nabla_2 S(\mathbf{x}, \mathbf{y})$ and $\mathbf{p}_2=\nabla_y S(\mathbf{x}, \mathbf{y})$, therefore become inextricably interdependent. The "quantum-mechanical" potential,

$$U=-\frac{\hbar^2}{2mR(\mathbf{x}, \mathbf{y})}(\nabla_x^2 R+\nabla_y^2 R)$$

ceases to be expressible as the sum of a term involving $\mathbf{x}$ and a term involving $\mathbf{y}$. The probability density,

[14] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1933).

$R^2(\mathbf{x}, \mathbf{y})$ can no longer be written as a product of a function of $\mathbf{x}$ and a function of $\mathbf{y}$, from which we conclude that the probability distributions of the two particles are no longer statistically independent. Moreover, the motion of the particle is exceedingly complicated, because the expressions for R and S are somewhat analogous to those obtained in the simpler problem of a nonstationary state of a single particle [see Eqs. (19) and (20)]. In the region where the scattered waves $\psi_n(\mathbf{x})f_n(\mathbf{y}, t)$ have an amplitude comparable with that of the incident wave, $\psi_0(\mathbf{x})f_0(\mathbf{y}, t)$, the functions R and S, and therefore the "quantum-mechanical" potential and the particle momenta, undergo rapid and violent fluctuations, both as functions of position and of time. Because the quantum-mechanical potential has $R(\mathbf{x}, \mathbf{y}, t)$ in the denominator, these fluctuations may become very large in this region where R is small. If the particles happen to enter such a region, they may exchange very large quantities of energy and momentum in a very short time, even if the classical potential, $V(\mathbf{x}, \mathbf{y})$ is very small. A small value of $V(\mathbf{x}, \mathbf{y})$ implies, however, a correspondingly small value of the scattered wave amplitudes, $f_n(\mathbf{y}, t)$. Since the fluctuations become large only in the region where the scattered wave amplitude is comparable with the incident wave amplitude and since the probability that the particles shall enter a given region of $\mathbf{x}, \mathbf{y}$ space is proportional to $R^2(\mathbf{x}, \mathbf{y})$, it is clear that a large transfer of energy is improbable (although still always possible) when $V(\mathbf{x}, \mathbf{y})$ is small.

While interaction between the two particles takes place then, their orbits are subject to wild fluctuations. Eventually, however, the behavior of the system quiets down and becomes simple again. For after the wave function takes its asymptotic form (26), and the packets corresponding to different values of n have obtained classically describable separations, we can deduce that because the probability density is $|\psi|^2$, the outgoing particle must enter one of these packets and stay with that packet thereafter (since it does not enter the space between packets in which the probability density is negligibly different from zero). In the calculation of the particle velocities, $\mathbf{V}_1=\nabla_x S/m$, $\mathbf{V}_2=\nabla_y S/m$, and of the quantum-mechanical potential, $U=(-\hbar^2/2mR)(\nabla_x{}^2R+\nabla_y{}^2R)$, we can therefore ignore all parts of the wave function other than the one actually containing the outgoing particle. It follows that the system acts as if it had the wave function

$$\Psi_n=\psi_n(\mathbf{x})\exp\left(\frac{iE_nt}{\hbar}\right)\int f(\mathbf{k}-\mathbf{k}_0)\times\frac{\exp\{i[\mathbf{k}_n\cdot\mathbf{r}-(\hbar k_n{}^2t/2m)t]\}}{r}g_n(\theta,\phi,\mathbf{k})d\mathbf{k}, \quad (29)$$

where n denotes the packet actually containing the outgoing particle. This means that for all practical purposes the complete wave function (26) of the system may be replaced by Eq. (29), which corresponds to an atomic electron in its nth quantum state, and to an outgoing particle with a correlated energy, $E_n'=\hbar^2k_n{}^2/2m$. Because the wave function is a product of a function of $\mathbf{x}$ and a function of $\mathbf{y}$, each system once again acts independently of the other. The wave function can now be renormalized because the multiplication of Ψ_n by a constant changes no physically significant quantity, such as the particle velocity or the "quantum-mechanical" potential. As shown in Sec. 5, when the electronic wave function is $\psi_n(\mathbf{x})\exp(-iE_nt/\hbar)$, its energy must be E_n. Thus, we have obtained a description of how it comes about that the energy is always transferred in quanta of size E_n-E_0.

It should be noted that while the wave packets are still separating, the electron energy is not quantized, but has a continuous range of values, which fluctuate rapidly. It is only the final value of the energy, appearing after the interaction is over that must be quantized. A similar result is obtained in the usual interpretation if one notes that because of the uncertainty principle, the energy of either system can become definite only after enough time has elapsed to complete the scattering process.[15]

In principle, the actual packet entered by the outgoing particle could be predicted if we knew the initial position of both particles and, of course, the initial form of the wave function of the combined system.[16] In practice, however, the particle orbits are very complicated and very sensitively dependent on the precise values of these initial positions. Since we do not at present know how to measure these initial positions precisely, we cannot actually predict the outcome of such an interaction process. The best that we can do is to predict the probability that an outgoing particle enters the nth packet within a given range of solid angle, $d\Omega$, leaving the hydrogen atom in its nth quantum state. In doing this, we use the fact that the probability density in $\mathbf{x}, \mathbf{y}$ space is $|\psi(\mathbf{x}, \mathbf{y})|^2$ and that as long as we are restricted to the nth packet, we can replace the complete wave function (26) by the wave function (29), corresponding to the packet that actually contains the particle. Now, by definition, we have $\int|\psi_n(\mathbf{x})|^2d\mathbf{x}=1$. The remaining integration of

$$\left|\int f(\mathbf{k}-\mathbf{k}_0)\frac{\exp\{i[k_nr-(\hbar k_n{}^2/2m)t]\}}{r}g_n(\theta,\phi,k)d\mathbf{k}\right|^2$$

over the region of space corresponding to the nth outgoing packet leads, however, to precisely the same probability of scattering as would have been obtained by applying the usual interpretation. We conclude, then, that if ψ satisfies Schroedinger's equation, that if $\mathbf{v}=\nabla S/m$, and that if the probability density of particles is $P(\mathbf{x}, \mathbf{y})=R^2(\mathbf{x}, \mathbf{y})$, we obtain in every respect

[15] See reference 2, Chapter 18, Sec. 19.

[16] Note that in the usual interpretation one assumes that *nothing* determines the precise outcome of an individual scattering process. Instead, one assumes that all descriptions are inherently and unavoidably statistical (see Sec. 2).

exactly the same physical predictions for this problem as are obtained when we use the usual interpretation.

There remains only one more problem; namely, to show that if the outgoing packets are subsequently brought together by some arrangement of matter that does not act on the atomic electron, the atomic electron and the scattered particle will continue to act independently.[17] To show that these two particles will continue to act independently, we note that in all practical applications, the outgoing particle soon interacts with some classically describable system. Such a system might consist, for example, of the host of atoms of the gas with which it collides or of the walls of a container. In any case, if the scattering process is ever to be observed, the outgoing particle must interact with a classically describable measuring apparatus. Now all classically describable systems have the property that they contain an enormous number of internal "thermodynamic" degrees of freedom that are inevitably excited when the outgoing particle interacts with the system. The wave function of the outgoing particle is then coupled to that of these internal thermodynamic degrees of freedom, which we represent as $y_1, y_2, \cdots y_s$. To denote this coupling, we write the wave function for the entire system as

$$\Psi=\sum_n \psi_n(\mathbf{x})\exp(-iE_nt/\hbar)f_n(\mathbf{y}, y_1, y_2\cdots y_s). \quad (30)$$

Now, when the wave function takes this form, the overlapping of different packets in $\mathbf{y}$ space is not enough to produce interference between the different $\psi_n(\mathbf{x})$. To obtain such interference, it is necessary that the packets $f_n(\mathbf{y}, y_1, y_2, \cdots y_s)$ overlap in every one of the $S+3$ dimensions, $\mathbf{y}, y_1, y_2\cdots y_s$. The reader will readily convince himself, by considering a typical case such as a collision of the outgoing particle with a metal wall, that it is overwhelmingly improbable that two of the packets $f_n(\mathbf{y}_1, y_1, y_2\cdots y_s)$ will overlap with regard to every one of the internal thermodynamic coordinates, $y_1, y_2, \cdots y_s$, even if they are successfully made to overlap in $\mathbf{y}$ space. This is because each packet corresponds to a different particle velocity and to a different time of collision with the metal wall. Because the myriads of internal thermodynamic degrees of freedom are so chaotically complicated, it is very likely that as each of the n packets interacts with them, it will encounter different conditions, which will make the combined wave packet $f_n(\mathbf{y}, y_1, \cdots y_s)$ enter very different regions of $y_1, y_2\cdots y_s$ space. Thus, for all practical purposes, we can ignore the possibility that if two of the packets are made to cross in $\mathbf{y}$ space, the motion either of the atomic electron or of the outgoing particle will be affected.[18]

[17] See reference 2, Chapter 22, Sec. 11, for a treatment of a similar problem.

[18] It should be noted that exactly the same problem arises in the usual interpretation of the quantum theory for (reference 16), for whenever two packets overlap, then even in the usual interpretation, the system must be regarded as, in some sense, covering the states corresponding to both packets simultaneously. See reference 2, Chapter 6 and Chapter 16, Sec. 25. Once two packets have obtained classically describable separations, then, both in the usual interpretation and in our interpretation the probability that there will be significant interference between them is so overwhelmingly small that it may be compared to the probability that a tea kettle placed on a fire will happen to freeze instead of boil. Thus, we may for all practical purposes neglect the possibility of interference between packets corresponding to the different possible energy states in which the hydrogen atom may be left.

8. PENETRATION OF A BARRIER

According to classical physics, a particle can never penetrate a potential barrier having a height greater than the particle kinetic energy. In the usual interpretation of the quantum theory, it is said to be able, with a small probability, to "leak" through the barrier. In our interpretation of the quantum theory, however, the potential provided by the Schroedinger ψ-field enables it to "ride" over the barrier, but only a few particles are likely to have trajectories that carry them all the way across without being turned around.

We shall merely sketch in general terms how the above results can be obtained. Since the motion of the particle is strongly affected by its ψ-field, we must first solve for this field with the aid of "Schroedinger's equation." Initially, we have a wave packet incident on the potential barrier; and because the probability density is equal to $|\psi(\mathbf{x})|^2$, the particle is certain to be somewhere within this wave packet. When the wave packet strikes the repulsive barrier, the ψ-field undergoes rapid changes which can be calculated[19] if desired, but whose precise form does not interest us here. At this time, the "quantum-mechanical" potential, $U=(-\hbar^2/2m)\nabla^2R/R$, undergoes rapid and violent fluctuations, analogous to those described in Sec. 7 in connection with Eqs. (19), (20), and (25). The particle orbit then becomes very complicated and, because the potential is time dependent, very sensitive to the precise initial relationship between the particle position and the center of the wave packet. Ultimately, however, the incident wave packet disappears and is replaced by two packets, one of them a reflected packet and the other a transmitted packet having a much smaller intensity. Because the probability density is $|\psi|^2$, the particle must end up in one of these packets. The other packet can, as shown in Sec. 7, subsequently be ignored. Since the reflected packet is usually so much stronger than the transmitted packet, we conclude that during the time when the packet is inside the barrier, most of the particle orbits must be turned around, as a result of the violent fluctuations in the "quantum-mechanical" potential.

9. POSSIBLE MODIFICATIONS IN MATHEMATICAL FORMULATION LEADING TO EXPERIMENTAL PROOF THAT NEW INTERPRETATION IS NEEDED

We have already seen in a number of cases and in Paper II we shall prove in general, that as long as we

[19] See, for example, reference 2, Chapter 11, Sec. 17, and Chapter 12, Sec. 18.

assume that ψ satisfies Schroedinger's equation, that $\mathbf{v}=\nabla S(\mathbf{x})/m$, and that we have a statistical ensemble with a probability density equal to $|\psi(\mathbf{x})|^2$, our interpretation of the quantum theory leads to physical results that are identical with those obtained from the usual interpretation. Evidence indicating the need for adopting our interpretation instead of the usual one could therefore come only from experiments, such as those involving phenomena associated with distances of the order of 10^{-13} cm or less, which are not now adequately understood in terms of the existing theory. In this paper we shall not, however, actually suggest any specific experimental methods of distinguishing between our interpretation and the usual one, but shall confine ourselves to demonstrating that such experiments are conceivable.

Now, there are an infinite number of ways of modifying the mathematical form of the theory that are consistent with our interpretation and not with the usual interpretation. We shall confine ourselves here, however, to suggesting two such modifications, which have already been indicated in Sec. 4, namely, to give up the assumption that $\mathbf{v}$ is necessarily equal to $\nabla S(\mathbf{x})/m$, and to give up the assumption that ψ must necessarily satisfy a homogeneous linear equation of the general type suggested by Schroedinger. As we shall see, giving up either of those first two assumptions will in general also require us to give up the assumption of a statistical ensemble of particles, with a probability density equal to $|\psi(\mathbf{x})|^2$.

We begin by noting that it is consistent with our interpretation to modify the equations of motion of a particle (8a) by adding any conceivable force term to the right-hand side. Let us, for the sake of illustration, consider a force that tends to make the difference, $\mathbf{p}-\nabla S(\mathbf{x})$, decay rapidly with time, with a mean decay time of the order of $\tau=10^{-13}/c$ seconds, where c is the velocity of light. To achieve this result, we write

$$m\frac{d^2\mathbf{x}}{dt^2}=-\nabla\left\{V(\mathbf{x})-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right\}+\mathbf{f}(\mathbf{p}-\nabla S(\mathbf{x})), \quad (31)$$

where $\mathbf{f}(\mathbf{p}-\nabla S(\mathbf{x}))$ is assumed to be a function which vanishes when $\mathbf{p}=\nabla S(\mathbf{x})$ and more generally takes such a form that it implies a force tending to make $\mathbf{p}-\nabla S(\mathbf{x})$ decrease rapidly with the passage of time. It is clear, moreover, that f can be so chosen that it is large only in processes involving very short distances (where $\nabla S(\mathbf{x})$ should be large).

If the correct equations of motion resembled Eq. (31), then the usual interpretation would be applicable only over times much longer than τ, for only after such times have elapsed will the relation $\mathbf{p}=\nabla S(\mathbf{x})$ be a good approximation. Moreover, it is clear that such modifications of the theory cannot even be described in the usual interpretation, because they involve the precisely definable particle variables which are not postulated in the usual interpretation.

Let us now consider a modification that makes the equation governing ψ inhomogeneous. Such a modification is

$$i\hbar\psi/\partial t=H\psi+\xi(\mathbf{p}-\nabla S(\mathbf{x}_i)). \quad (32)$$

Here, H is the usual Hamiltonian operator, $\mathbf{x}_i$, represents the actual location of the particle, and ξ is a function that vanishes when $\mathbf{p}=\nabla S(\mathbf{x}_i)$. Now, if the particle equations are chosen, as in Eq. (31), to make $\mathbf{p}-\nabla S(\mathbf{x}_i)$ decay rapidly with time, it follows that in atomic processes, the inhomogeneous term in Eq. (32) will become negligibly small, so that Schroedinger's equation is a good approximation. Nevertheless, in processes involving very short distances and very short times, the inhomogeneities would be important, and the ψ-field would, as in the case of the electromagnetic field, depend to some extent on the actual location of the particle.

It is clear that Eq. (32) is inconsistent with the usual interpretation of the theory. Moreover, we can contemplate further generalizations of Eq. (32), in the direction of introducing nonlinear terms that are large only for processes involving small distances. Since the usual interpretation is based on the hypothesis of linear superposition of "state vectors" in a Hilbert space, it follows that the usual interpretation could not be made consistent with such a nonlinear equation for a one-particle theory. In a many-particle theory, operators can be introduced, satisfying a nonlinear generalization of Schroedinger's equation; but these must ultimately operate on wave functions that satisfy a linear homogeneous Schroedinger equation.

Finally, we repeat a point already made in Sec. 4, namely, that if the theory is generalized in any of the ways indicated here, the probability density of particles will cease to equal $|\psi(\mathbf{x})|^2$. Thus, experiments would become conceivable that distinguish between $|\psi(\mathbf{x})|^2$ and this probability; and in this way we could obtain an experimental proof that the usual interpretation, which gives $|\psi(\mathbf{x})|^2$ *only* a probability interpretation, must be inadequate. Moreover, we shall show in Paper II that modifications like those suggested here would permit the particle position and momentum to be measured simultaneously, so that the uncertainty principle could be violated.

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