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THE DEDUCTION THEOREM IN A FUNCTIONAL CALCULUS OF FIRST ORDER BASED ON STRICT IMPLICATION

RUTH C. BARCAN

In a previous paper,[1] a functional calculus based on strict implication was developed. That system will be referred to as $S2^1$. The system resulting from the addition of Becker's[2] axiom "$\Diamond\Diamond A \mathrel{-\!\!3} \Diamond A$" will be referred to as $S4^1$. In the present paper[3] we will show that a restricted deduction theorem is provable in $S4^1$ or more precisely in a system equivalent to $S4^1$. We will also show that such a deduction theorem is not provable in $S2^1$.

The following theorems not derived in *Symbolic logic* will be required for the fundamental theorems XXVIII* and XXIX* of this paper. We will state most of them without proofs.

96. $\vdash (A \supset (B \supset E)) \mathrel{-\!\!3} ((A \supset B) \supset (A \supset E))$

97. $\vdash ((A \supset B) \supset (A \supset E)) \mathrel{-\!\!3} (A \supset (B \supset E))$

98. $\vdash (A \supset (B \supset E)) \boldsymbol{\equiv} ((A \supset B) \supset (A \supset E))$

99. $\vdash (A \supset (B \equiv E)) \boldsymbol{\equiv} ((A \supset B) \equiv (A \supset E))$

$((A \supset (B \supset E))(A \supset (E \supset B))) \boldsymbol{\equiv} (((A \supset B) \supset (A \supset E))((A \supset E) \supset (A \supset B)))$ 98, adj, 80, mod pon

$(A \supset (B \equiv E)) \boldsymbol{\equiv} ((A \supset B) \equiv (A \supset E))$ *16.8*, subst, def

100. $\vdash A \supset A$

101. $\vdash ((A \supset B)(A \supset (B \mathrel{-\!\!3} E))) \mathrel{-\!\!3} (A \supset E)$

XXV. If $\vdash A \mathrel{-\!\!3} B$ then $\vdash (AE) \mathrel{-\!\!3} B$.

XXVI. If $\vdash E \mathrel{-\!\!3} (A \equiv B)$ then $\vdash ((H \supset E)(H \supset A)) \mathrel{-\!\!3} (H \supset B)$ and $\vdash ((H \supset E)(H \supset B)) \mathrel{-\!\!3} (H \mathrel{-\!\!3} A)$.

$(\sim H \vee E) \mathrel{-\!\!3} (\sim H \vee (A \equiv B))$ hyp, *19.64*, mod pon, *13.11*, subst

$(H \supset E) \mathrel{-\!\!3} ((H \supset A) \equiv (H \supset B))$ *14.2*, 99, subst, def, 2, VIII

$((H \supset E)(H \supset A)) \mathrel{-\!\!3} (H \supset B))$ *14.26*, subst

Similarly,

$((H \supset E)(H \supset B)) \mathrel{-\!\!3} (H \supset A)$

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[1] *A functional calculus of first order based on strict implication*, this JOURNAL, vol. 11 (1946), pp. 1–16.

[2] See Lewis and Langford, *Symbolic logic*, pp. 497–502.

[3] Part of this paper was included in a dissertation written in partial fulfillment of the requirements for the Ph.D. degree in Philosophy at Yale University. I am grateful to Professor Frederic B. Fitch for his criticisms and suggestions.

102. $\vdash (H \prec (A \equiv B)) \prec ((H \prec A) \supset (H \prec B))$ and
$\vdash (H \prec (A \equiv B)) \prec ((H \prec B) \supset (H \prec A))$.

XXVII. If $\vdash E \prec (A \equiv B)$ then $\vdash ((H \prec E)(H \prec A)) \prec (H \prec B)$ and
$\vdash ((H \prec E)(H \prec B)) \prec (H \prec A)$.

$(\sim H \vee E) \prec (\sim H \vee (A \equiv B))$ hyp, *19.64*, mod pon, *13.11*, subst
$(H \prec E) \prec (H \prec (A \equiv B))$ *14.2*, VII, *18.7*, subst
$((H \prec E)(H \prec A)) \prec (H \prec B)$ 102, VIII, *14.26*, subst

Similarly,
$((H \prec E)(H \prec B)) \prec (H \prec A)$.

The axiom which distinguishes $S4^1$ is 103*. Theorems derivable in $S4^1$ but not in $S2^1$ will be marked by an asterisk.

104* and 105* are required in the proof of XIX*.

103*. $\vdash \Diamond\Diamond A \prec \Diamond A$
104*. $\vdash \Box\Box A \equiv \Box A$
105*. $\vdash \Box A \prec (B \prec \Box A)$

$S2^1$eq and $S4^1$eq. A consideration of the deduction theorem requires a definition on "proof on hypotheses." Such a definition is facilitated if we formulate it in terms of a system equivalent to $S2^1$ which will be referred to as $S2^1$eq.

Every axiom of $S2^1$ is an axiom of $S2^1$eq. The rule for generalization in $S2^1$ is replaced by the following rule for axioms: If A is an axiom then (β)B is an axiom where B is the result of replacing all free occurrences of α in A by β. The rule for adjunction is like that of $S2^1$ extended to include the following: If $(\alpha_1)(\alpha_2) \cdots (\alpha_m)$A and $(\alpha_1)(\alpha_2)$B $\cdots (\alpha_m)$ are provable then $(\alpha_1)(\alpha_2) \cdots (\alpha_m)$(AB) is provable. The substitution rule of $S2^1$ is extended so as to read exactly like XVI. Modus ponens is retained.

The rule for axioms gives the effect of generalization since we can prove the following: If $A_1, A_2, \cdots, A_n$ are the steps of a proof of B where B is A_n then we can construct a corresponding proof such that (α)B is provable. Suppose A_i is an axiom, then replace A_i by $(\alpha)A_i$. If A_i is not an axiom then it follows from some previous A_{i_1} and A_{i_2} by modus ponens, adjunction or substitution. Suppose A_i follows by modus ponens. Let A_{i_2} be $A_{i_1} \prec A_i$. One of the theorems derivable in $S2^1$eq is $(\alpha)(A \prec B) \prec ((\alpha)A \prec (\alpha)B)$ the proof of which is the same as 19 of $S2^1$ since the rule of generalization is not employed. Replace A_i by the sequence $(\alpha)(A_{i_1} \prec A_i) \prec ((\alpha)A_{i_1} \prec (\alpha)A_i)$, $(\alpha)A_{i_1} \prec (\alpha)A_i$, $(\alpha)A_i$. If A_i follows from some preceding A_{i_1} and A_{i_2} by substitution or adjunction then replace A_i by $(\alpha)A_i$.

It is obvious that $S2^1$ is equivalent to $S2^1$eq. The axioms of $S2^1$ and the generalization rule give us the axioms of $S2^1$eq. Modus ponens is retained. The extended adjunction rule follows directly from 29 and modus ponens. XVI is the same as the extended rule of substitution.

$S4^1$eq is the system which results from the addition of axiom 103* to $S2^1$eq and it is of course equivalent to $S4^1$.

Proof on hypotheses. Let B be said to be provable on the hypotheses A_1, $A_2, \cdots, A_n$ in $S2^1$eq and $S4^1$eq if there is a finite list of formulas $B_1, B_2, \cdots, B_s$ where B_s is B, satisfying the following conditions:

For each i $(1 \leqq i \leqq s)$

1. B_i is one of $A_1, A_2, \cdots, A_n$ or
2. B_i is an axiom or
3. B_i results by one of the rules of inference from B_{i_1} and B_{i_2} where $i_1 < i$ and $i_2 < i$.

B is provable on the hypotheses $A_1, A_2, \cdots, A_n$ will be abbreviated: $A_1, A_2, \cdots, A_n \vdash B$.

In $S2^1eq$ we cannot prove either

1. $A_1, A_2, \cdots, A_{n-1} \vdash A_n \supset B$

or 2. $A_1, A_2, \cdots, A_{n-1} \vdash A_n \prec B$

from 3. $A_1, A_2, \cdots, A_n \vdash B$.

This can be shown if we use an eight element matrix of Parry[4] which satisfies the axioms and rules of S2. This matrix also satisfies $S2^1eq$ if we regard the domain of individuals as consisting of a single individual a.[5] Every expression of the form $(\alpha)A$ would then be replaced by B where B results from substituting all free occurrences of α in A by a. Neither $(A \prec B) \supset (\Diamond A \prec \Diamond B)$ nor $(A \prec B) \prec (\Diamond A \prec \Diamond B)$ are satisfied by this matrix although $(\Diamond A \prec \Diamond B)$ is provable on the hypothesis $(A \prec B)$ in $S2^1eq$. (Rule VI.)

In $S4^1eq$, 1 always follows from 3 and 2 follows from 3 if each A_r $(1 \leqq r \leqq n)$ can be transformed into an equivalent expression $\Box\Gamma$.

XXVIII*. If $A_1, A_2, \cdots, A_n \vdash B$ then $A_1, A_2, \cdots, A_{n-1} \vdash A_n \supset B$.

Proof: Let us assume that $A_n \supset B_m$ has been proved for every B_m in the list $B_1, B_2, \cdots, B_s$ of the definition of proof on hypotheses where $m < i$. We will show that $\vdash A_n \supset B_i$.

Case (1). B_i is an axiom.

$B_i \prec (A_n \supset B_i)$ *15.2*

$A_n \supset B_i$ mod pon

Case (2). B_i is A_n.

$A_n \supset B_i$ 100

Case (3). B_i is one of $A_1, A_2, \cdots, A_{n-1}$.

Proof like Case (1).

Case (4). B_i follows by modus ponens from a previous B_{i_1} and B_{i_2} where let us say B_{i_2} is $B_{i_1} \prec B_i$.

$((A_n \supset B_{i_1})(A_n \supset (B_{i_1} \prec B_i))) \prec (A_n \supset B_i)$ 101

$(A_n \supset B_{i_1})(A_n \supset (B_{i_1} \prec B_i))$ hyp, adj

$(A_n \supset B_i)$ mod pon

Case (5). B_i follows from adjunction of a previous B_{i_1} and B_{i_2}.

$((A_n \supset B_{i_1})(A_n \supset B_{i2})) \prec (A_n \supset (B_{i_1}B_{i_2}))$

16.8, def, *12.17*, mod pon

Where the extended rule is used we have

$((A_n \supset (\alpha_1)(\alpha_2) \cdots (\alpha_m)B_{i_1})(A_n \supset (\alpha_1)(\alpha_2) \cdots (\alpha_m)B_{i_2})) \prec (A_n \supset (\alpha_1)(\alpha_2) \cdots (\alpha_m)(B_{i_1}B_{i_2}))$ Like step 1 using 29 and subst.

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[4] W. T. Parry, *The postulates for "strict implication,"* ***Mind,*** vol. 43 (1934), pp. 78–80.

[5] This method for interpreting the quantifiers was suggested by the referee.

$A_n \supset B_i$ hyp, adj, mod pon

Case (6). B_i follows by substitution from a previous B_{i_1} and B_{i_2} where let us say B_{i_2} is $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$ and $\alpha_1, \alpha_2, \cdots, \alpha_m$ is a complete list of the free variables in Γ and E.[6]

$(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \dashv (B_{i_1} \equiv B_i)$ XIX*, *14.1*, IX, VIII

$((A_n \supset (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E))(A_n \supset B_{i_1})) \dashv (A_n \supset B_i)$ XXVI

$A_n \supset B_i$ hyp, adj. mod pon

XXIX*. If $A_1, A_2, \cdots, A_n \vdash B$ and if $\vdash A_1 \equiv \Box\Gamma_1$, $\vdash A_2 \equiv \Box\Gamma_2, \cdots,$ $\vdash A_n \equiv \Box\Gamma_n$, then $A_1, A_2, \cdots, A_{n-1} \vdash A_n \dashv B$.

Proof: Let us assume that $A_n \dashv B_m$ has been proved for every B_m in the list $B_1, B_2, \cdots, B_s$ of the definition of proof on hypotheses where $m < i$. We will show that $\vdash A_n \dashv B_i$.

Case (1). B_i is an axiom. Since every axiom of $S4^1$eq is of the form $E \dashv H$ or $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(E \dashv H)$ it follows from *18.7*, 39 and substitution that if M is an axiom then $\vdash M \equiv \Box\Gamma$.

$B_i \dashv (A_n \dashv B_i)$ 105*

$A_n \dashv B_i$ mod pon

Case (2). B_i is A_n

$A_n \dashv B_i$ *12.1*

Case (3). B_i is one of $A_1, A_2, \cdots, A_n$

$A_n \dashv B_i$ 105*, hyp, mod pon

Case (4). B_i follows from a previous B_{i_1} and B_{i_2} by modus ponens where let us say B_{i_2} is $B_{i_1} \dashv B_i$.

$((A_n \supset B_{i_1})(A_n \supset (B_{i_1} \dashv B_i))) \dashv (A_n \supset B_i)$ 101

$((A_n \dashv B_{i_1})(A_n \dashv (B_{i_1} \dashv B_i))) \dashv (A_n \dashv B_i)$ VII, *19.81*, *18.7*, subst

$(A_n \dashv B_{i_1})(A_n \dashv (B_{i_1} \dashv B_i))$ hyp, adj

$A_n \dashv B_i$ mod pon

Case (5). B_i follows from adjunction of a previous B_{i_1} and B_{i_2}.

$((A_n \dashv B_{i_1})(A_n \dashv B_{i_2})) \dashv (A_n \dashv (B_{i_1}B_{i_2}))$ *19.61*

$A_n \dashv B_i$ hyp, adj, mod pon

Where the extended rule is used employ 29 and substitution as in XXVIII*.

Case (6). B_i follows from a previous B_{i_1} and B_{i_2} by substitution where let us say B_{i_2} is $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$ and $\alpha_1, \alpha_2, \cdots, \alpha_m$ is a complete list of the free variables in Γ and E.[6]

$(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \dashv (B_{i_1} \equiv B_i)$ XIX*

$((A_n \dashv (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E))(A_n \dashv B_{i_1})) \dashv (A_n \dashv B_i)$ XXVII

$A_n \dashv B_i$ hyp, adj. mod pon[7]

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[6] If $\Gamma \equiv E$ is provable then $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$ is provable where $\alpha_1, \alpha_2, \cdots, \alpha_m$ is a complete list of the free variables in Γ and E. We will assume that wherever B_i follows by substitution, the variables in Γ and E have been generalized upon.

[7] A slightly stronger theorem than XXIX* could be proved as an immediate corollary of XXVIII* if we introduced the following lemma: If $\vdash A \supset B$ then $\vdash \Box A \supset \Box B$. We would then need only to assume that $\vdash A_n \equiv \Box\Gamma_i$ where $i < n$. This alternative proof was suggested by the referee.