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Relativity Quantum Mechanics with an Application to Compton Scattering.

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§ 1. *Introduction.*

The new quantum mechanics, introduced by Heisenberg* and since developed from different points of view by various authors,† takes its simplest form if one assumes merely that the dynamical variables are numbers of a special type (called q-numbers to distinguish them from ordinary or c-numbers) that obey all the ordinary algebraic laws except the commutative law of multiplication, and satisfy instead of this the relations

$$\left.\begin{aligned} q_r q_s - q_s q_r = 0, \qquad p_r p_s - p_s p_r = 0 \\ q_r p_s - p_s q_r = 0 \ (r \neq s) \text{ or } ih (r = s) \end{aligned}\right\} \tag{1}$$

where the p's and q's are a set of canonical variables and h is a c-number equal to $(2\pi)^{-1}$ times the usual Planck's constant. Equations (1) may be regarded as replacing the commutative law of the classical theory, as one can, with their help, build up a complete algebraic theory of quantities that are analytic functions of a set of canonical variables. Further, it may easily be seen that the quantity $[x, y]$ defined by

$$xy - yx = ih[x, y] \tag{2}$$

is completely analogous to the Poisson bracket of the classical theory. By means of this analogy the whole of the classical dynamical theory, in so far as it can be expressed in terms of P.B.'s instead of differential coefficients, may be taken over immediately into the quantum theory.

It has been shown by the author‡ that the quantum solution of a multiply periodic dynamical system may be effected, as on the classical theory, by the introduction of uniformising variables, J's and w's, and the results can then

* Heisenberg, 'Zeits. f. Phys.,' vol. 33, p. 879 (1925).

† Born and Jordan, 'Zeits. f. Phys.,' vol. 34, p. 858 (1925); Born, Heisenberg and Jordan, 'Zeits. f. Phys.,' vol. 35, p. 557 (1926); Kramers, 'Physica,' vol. 5, p. 369 (1925); Dirac, 'Roy. Soc. Proc.,' A, vol. 109, p. 642 (1925); Born and Wiener, 'Zeits. f. Phys.,' vol. 36, p. 174 (1926) *or* 'Jour. Math. Phys. Mass.,' vol. 5, p. 84 (1926).

‡ 'Roy. Soc. Proc.,' A, vol. 110, p. 561 (1926).

be interpreted in a way of which the following is a brief outline. The total polarisation of the system can be expanded as a Fourier series in the w's whose coefficients are functions of the J's only. On the classical theory, if one takes one of these coefficients, say, that of $e^{i(\alpha w)}$ where $(\alpha w) = \Sigma \alpha_r w_r$ and the α's are integers, and substitutes in it for the J_r a set of numbers, κ_r, say, the number thus obtained will determine the intensity of the $e^{i(\alpha w)}$ component of the radiation emitted by the system when in the state fixed by the equations $J_r = \kappa_r$. On the quantum theory, however, an ambiguity arises, since in the Fourier expansion of the polarisation the coefficients may be either in front of or behind their respective exponentials. The $e^{i(\alpha w)}$ term, for instance, would be

$$\tfrac{1}{2} C_\alpha e^{i(\alpha w)} = \tfrac{1}{2} e^{i(\alpha w)} C_\alpha',$$

where C_α and C_α' are in general two different functions of the J's, so that if one substitutes for the J_r the values κ_r, where the κ's are a set of c-numbers that may be regarded as fixing a stationary state of the system, one would obtain two $e^{i(\alpha w)}$ intensities related to this state. If, now, one puts

$$J_r e^{i(\alpha w)} = e^{i(\alpha w)} J_r',$$

then C_α must be the same function of the J's that C_α' is of the J′'s, so that if one substituted for the J_r in C_α the values κ_r, one would obtain the same result (a c-number, of course) as if one substituted for the J_r in C_α' their values given by the equations $J_r' = \kappa_r$, and one may therefore suppose this result to determine the intensity of a component of the emitted radiation that is symmetrically related to the two states of the system given by $J_r = \kappa_r$ and $J_r' = \kappa_r$. It may be shown that $J_r' = J_r + \alpha_r h$, and hence the two states are respectively the initial and final states on Bohr's theory. It may also be shown that the system has transition frequencies related to pairs of states as on Bohr's theory.

It now remains only to determine what values one shall assume the κ's to take, and this may require an appeal to physical considerations. For the case of the simple harmonic oscillator it has been shown rigorously by Born and Jordan* that the action variable can take only a certain discrete set of values, one of which gives a state of lowest energy, and their method seems to be capable of extension. For the case of Compton scattering by a free electron, considered in the present paper, there is no restriction on the values that the action variable can take. The initial value of the action variable is now determined by the initial velocity of the electron, which must, of course, be given from physical considerations.

It will be observed that the notion of canonical variables plays a very funda-

* Born and Jordan, *loc. cit.*, § 5.

mental part in the theory. Any attempt to extend the domain of the present quantum mechanics must be preceded by the introduction of canonical variables into the corresponding classical theory, with a reformulation of this classical theory with P.B.'s instead of differential coefficients. The object of the present paper is to obtain in this way the extension of the quantum mechanics to systems for which the Hamiltonian involves the time explicitly (§ 2) and to relativity mechanics (§§ 3, 4).

§ 2. *Quantum Time.*

Consider a dynamical system of u degrees of freedom for which the Hamiltonian H involves the time explicitly. The principle of relativity demands that the time shall be treated on the same footing as the other variables, and so it must therefore be a q-number. On the classical theory it is known that one may solve the problem by considering the time t to be an extra co-ordinate of the system, with minus the energy (or perhaps a slightly different quantity) W as conjugate momentum. In the solution of the problem there will now be complete symmetry between the new pair of variables t and $-\mathrm{W}$ and the original u pairs, except for the fact that when one performs the contact transformation to the uniformising variables, the co-ordinate t itself must be one of the new variables. A P.B. is now defined by

$$[x, y] = \Sigma_r \left(\frac{\partial x}{\partial q_r}\frac{\partial y}{\partial p_r} - \frac{\partial x}{\partial p_r}\frac{\partial y}{\partial q_r}\right) - \frac{\partial x}{\partial t}\frac{\partial y}{\partial \mathrm{W}} + \frac{\partial x}{\partial \mathrm{W}}\frac{\partial y}{\partial t}, \tag{3}$$

and is invariant under any contact transformation of the $(2u + 2)$ variables. A dynamical system is now determined by an *equation* between the $(2u + 2)$ variables instead of a *function* of $2u$ variables—namely, the Hamiltonian equation

$$\mathrm{H} - \mathrm{W} = 0, \tag{4}$$

and the equations of motion are

$$\left.\begin{aligned} \dot{q}_r &= \frac{\partial \mathrm{H}}{\partial p_r} = \frac{\partial (\mathrm{H} - \mathrm{W})}{\partial p_r} \\ \dot{t} &= \ 1 \ = \frac{\partial (\mathrm{H} - \mathrm{W})}{\partial (-\mathrm{W})} \\ \dot{p}_r &= \frac{\partial \mathrm{H}}{\partial q_r} = -\frac{\partial (\mathrm{H} - \mathrm{W})}{\partial q_r} \end{aligned}\right\}, \tag{5}$$

and lastly

$$\begin{aligned} -\dot{\mathrm{W}} = -\dot{\mathrm{H}} &= -\Sigma_r \left(\frac{\partial \mathrm{H}}{\partial q_r}\dot{q}_r + \frac{\partial \mathrm{H}}{\partial p_r}\dot{p}_r\right) - \frac{\partial \mathrm{H}}{\partial t} = -\frac{\partial \mathrm{H}}{\partial t}. \\ &= -\frac{\partial (\mathrm{H} - \mathrm{W})}{\partial t}. \end{aligned} \tag{5A}$$

From these equations of motion, if x is any function of the $(2u + 2)$ variables

$$\dot{x} = \Sigma_r \left(\frac{\partial x}{\partial q_r} \dot{q}_r + \frac{\partial x}{\partial p_r} \dot{p}_r\right) + \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \mathrm{W}} \dot{\mathrm{W}}$$

$$= \Sigma_r \left(\frac{\partial x}{\partial q_r} \frac{\partial (\mathrm{H} - \mathrm{W})}{\partial p_r} - \frac{\partial x}{\partial p_r} \frac{\partial (\mathrm{H} - \mathrm{W})}{\partial q_r}\right) - \frac{\partial x}{\partial t} \frac{\partial (\mathrm{H} - \mathrm{W})}{\partial \mathrm{W}} + \frac{\partial x}{\partial \mathrm{W}} \frac{\partial (\mathrm{H} - \mathrm{W})}{\partial t}$$

or

$$\dot{x} = [x, \mathrm{H} - \mathrm{W}] \tag{6}$$

from (3).

We can take these results directly over into the quantum theory. We assume that t and $-\mathrm{W}$ are a new pair of conjugate variables, and therefore satisfy the equations, supplementary to (1),

$$\left.\begin{array}{ll} tq_r - q_r t = 0, & tp_r - p_r t = 0 \\ \mathrm{W}q_r - q_r\mathrm{W} = 0, & \mathrm{W}p_r - p_r\mathrm{W} = 0 \\ \multicolumn{2}{c}{t\mathrm{W} - \mathrm{W}t = -ih} \end{array}\right\}, \tag{7}$$

and that the quantum P.B. $[x, y]$, defined by (2), is now the analogue of the classical expression on the right-hand side of (3). The equations of motion are assumed to be still given by (6).

The fact that a dynamical system is now specified by a Hamiltonian equation $\mathrm{H} - \mathrm{W} = 0$ instead of by a Hamiltonian function H here leads to a difficulty, since the Hamiltonian equation is not consistent with the quantum conditions (1) and (7). For example, if x is a function of the p's and q's only,

$$x\mathrm{W} - \mathrm{W}x = 0,$$

while in general

$$x\mathrm{H} - \mathrm{H}x \neq 0,$$

and these two equations are not consistent with $\mathrm{W} = \mathrm{H}$. An ordinary quantum equation gives a correct result when one equates the P.B. of either side with an arbitrary quantity, and must therefore correspond to an identity on the classical theory, *i.e.*, a relation that remains true on being differentiated partially with respect to any of the canonical variables. Now the Hamiltonian equation on the classical theory is not an identity. One can perform algebraic operations upon it, but one must not differentiate it. There must be a corresponding restriction on the use of the quantum Hamiltonian equation, although it cannot easily be specified, as there is no hard-and-fast distinction between algebraic operations and differentiations on the quantum theory. This uncertainty does not give any trouble in the present paper, however, as we shall follow the classical theory so closely that it will be immediately obvious whether any quantum operation corresponds to a legitimate classical operation or not.

The rules for the solution of the problem on the quantum theory are now, as on the classical theory, that one must determine a set of $(2u + 2)$ uniformising variables $J_0 \dots J_u$, $W_0 \dots W_u$, say, that satisfy the following conditions :—

(i) They must be canonical variables, it being possible to verify this without the use of the Hamiltonian equation.

(ii) One of the w's, w_0 say, must be just t.

(iii) The Hamiltonian equation must become a relation between the J's only.

(iv) The original variables, when expressed in terms of the new variables, must be multiply periodic functions of as many of the w's as possible with the periods 2π. They cannot, of course, be periodic functions of w_0, since $t = w_0$.

The frequencies associated with the various transitions of the system and the corresponding intensities may now be determined as for systems for which the Hamiltonian does not contain the time explicitly.

The fact that $w_0 = t$ provides us with certain information concerning the form of the transformation to the uniformising variables, as on the classical theory. Since each of the uniformising variables except J_0 commutes with w_0, *i.e.*, with t, when expressed in terms of the original variables, it must be independent of W. Further, since

$$[t, J_0] = [w_0, J_0] = 1 = -[t, W],$$

$J_0 + W$ commutes with t, and hence J_0, when expressed in terms of the origina variables, must equal minus W plus a quantity independent of W. The Hamiltonian equation $H - W = 0$ thus takes the form $H_0 + J_0 = 0$, where H_0 is a function of $J_1 \dots J_u$ only. In consequence of these results and the fact that t commutes with each of the p's and q's, Born, Heisenberg, and Jordan's perturbation theory for systems for which the Hamiltonian contains the time explicitly,* in which t is treated as a c-number, can be justified.

It should be observed that if the Hamiltonian equation of a system is $F(p_r, q_r, W, t) = 0$, it must be put in the standard form (4) before one can insert its left-hand side in the P.B. in the equation of motion (6). If one does not do this, but simply takes for the right-hand side of (6) the P.B. $[x, F]$, on the classical theory, the left-hand side would not be $\dot{x}$ but dx/dv, where v might be any variable. Also, with regard to condition (iii) for the uniformising variables, the quantity $H - W$ becomes just the quantity $H_0 + J_0$, but the quantity F may not become a function of the J's only, as one may have to divide the equation $F = 0$ by a factor which is a function of the w's as well as the J's in order to make its left-hand side a function of the J's only.

* Born, Heisenberg, and Jordan, *loc. cit.*, Kap. 1, § 5.

§3. *Quantum Mechanics of Moving Systems.*

A dynamical system that is moving as a whole may be described with, for canonical variables, the Cartesian co-ordinates of the centre of gravity x_1, x_2, x_3, with p_1, p_2, p_3, the components of total momentum, for conjugate variables, together with the necessary internal variables, which are independent of the position and velocity of the centre of gravity. If t is the time and W the energy, one may introduce the variables

$$x_4 = ict, \qquad p_4 = i\mathrm{W}/c, \tag{8}$$

where i is a root of -1 independent of the root of -1 occurring in the quantum conditions, and c is the velocity of light, which is, of course, a c-number. The principle of relativity requires complete symmetry between the x_4, p_4 and the x_1, p_1, the x_2, p_2, and the x_3, p_3. Hence, on account of the relations

$$[x_1, p_1] = [x_2, p_2] = [x_3, p_3] = 1,$$

we must have

$$[x_4, p_4] = 1$$

which gives

$$[ict, i\mathrm{W}/c] = 1$$

or

$$[t, \mathrm{W}] = -1.$$

The principle of relativity thus shows that $-$ W is the momentum conjugate to t, in agreement with the results of the preceding §. The remaining ones of the quantum conditions (7) may be likewise obtained.

Let m be the rest-mass of the system, so that mc^2 is its proper energy. Then m and mc^2 are functions of the internal variables only, or, when the system consists of a single particle only, so that there are no internal variables, they are c-numbers. We have

$$\mathrm{W}^2/c^2 - p_1{}^2 - p_2{}^2 - p_3{}^2 = m^2c^2 \tag{9}$$

which is the Hamiltonian equation for the system. The variables p_1, p_2, p_3, W and x_1, x_2, x_3, t may be taken to be uniformising variables, as they satisfy all the conditions for this except the multiply periodic conditions for the x's, which they obviously cannot be expected to satisfy. The remaining uniformising variables will be functions of the internal variables only.

The theory may be extended to systems acted upon by external fields of force, provided the classical equations of motion can be put in the Hamiltonian form. Suppose, for instance, that the system possesses a total charge e (a c-number), considered to be concentrated at its centre of gravity, and is in an electromagnetic field describable by the vector potential κ_1, κ_2, κ_3 and the scalar potential ϕ, these four quantities being given functions of x_1, x_2, x_3 and t. Instead of ϕ we may use the quantity

$$\kappa_4 = i\phi,$$

analogous to the x_4 and p_4 introduced by equation (8), so that κ_1, κ_2, κ_3, κ_4 are the components of a 4-vector. On the classical theory the equations of motion of the centre of gravity of the system may be written, if one uses the summation convention of the tensor calculus,

$$\frac{d}{ds}\left(m\frac{dx_\mu}{ds}\right) = \frac{e}{c}\left(\frac{\partial\kappa_\nu}{\partial x_\mu} - \frac{\partial\kappa_\mu}{\partial x_\nu}\right)\frac{dx_\nu}{ds} \qquad (\mu, \nu = 1 \dots 4)$$

$$= \frac{e}{c}\frac{\partial\kappa_\nu}{\partial x_\mu}\frac{dx_\nu}{ds} - \frac{e}{c}\frac{d\kappa_\mu}{ds}, \tag{10}$$

where s is the proper time defined by

$$ds^2 = c^2\, dx_\mu\, dx_\mu.$$

Now define p_μ by

$$p_\mu = m\frac{dx_\mu}{ds} + \frac{e}{c}\kappa_\mu \qquad (\mu = 1 \dots 4), \tag{11}$$

instead of simply by $m\, dx_\mu/ds$, which was its previous meaning. The equations of motion (10) become

$$\frac{dp_\mu}{ds} = \frac{e}{mc}\frac{\partial\kappa_\nu}{\partial x_\mu}\left(p_\nu - \frac{e}{c}\kappa_\nu\right). \tag{12}$$

The Hamiltonian equation (9) now becomes, owing to the changed meaning of the p's

$$-\left(p_\nu - \frac{e}{c}\kappa_\nu\right)\left(p_\nu - \frac{e}{c}\kappa_\nu\right) = m^2c^2 \tag{13}$$

or

$$\mathrm{F} = 0$$

where

$$\mathrm{F} = \frac{1}{2m}\left(p_\nu - \frac{e}{c}\kappa_\nu\right)\left(p_\nu - \frac{e}{c}\kappa_\nu\right) + \tfrac{1}{2}mc^2,$$

while the equations (11) and (12) may be written

$$\frac{dx_\nu}{ds} = \frac{\partial \mathrm{F}}{\partial p_\nu}$$

and

$$\frac{dp_\mu}{ds} = -\frac{\partial \mathrm{F}}{\partial x_\mu}.$$

Further, we have

$$\frac{\partial \mathrm{F}}{\partial m} = -\frac{1}{2m^2}\left(p_\nu - \frac{e}{c}\kappa_\nu\right)\left(p_\nu - \frac{e}{c}\kappa_\nu\right) + \tfrac{1}{2}c^2 = c^2$$

with the help of the equation $F = 0$, so that if q_r, p_r are a pair of canonically conjugate internal variables, the equations of internal motion are

$$\frac{dq_r}{ds} = \frac{\partial mc^2}{\partial p_r} = \frac{\partial m}{\partial p_r} c^2 = \frac{\partial m}{\partial p_r}\frac{\partial F}{\partial m} = \frac{\partial F}{\partial p_r}$$

$$\frac{dp_r}{ds} = -\frac{\partial mc^2}{\partial q_r} = -\frac{\partial m}{\partial q_r} c^2 = -\frac{\partial m}{\partial q_r}\frac{\partial F}{\partial m} = -\frac{\partial F}{\partial q_r}.$$

All the equations of motion are thus of the Hamiltonian form (5) with the Hamiltonian function F. The fact that the total differentiations are performed with respect to s instead of t is due to the Hamiltonian equation $F = 0$ not being in the standard form (4).

It is thus established that the classical equations of motion take the canonical form when the variables conjugate to the x_μ are defined by (11).* On the quantum theory we must therefore still use this definition of p_μ, and can then proceed according to rule with the Hamiltonian equation (13).

The κ's on the classical theory must satisfy the conditions

$$\frac{\partial \kappa_\mu}{\partial x_\mu} = 0, \qquad \frac{\partial^2 \kappa_\mu}{\partial x_\nu \partial x_\nu} = 0.$$

These equations may be written

$$[\kappa_\mu, p_\mu] = 0, \qquad [[\kappa_\mu, p_\nu], p_\nu] = 0, \tag{14}$$

and can then be taken over into the quantum theory. With the help of the first of these relations, the Hamiltonian equation (13) may be put in the forms

$$-m^2c^2 = p_\nu p_\nu - 2\frac{e}{c} p_\nu \kappa_\nu + \frac{e^2}{c^2}\kappa_\nu\kappa_\nu = p_\nu p_\nu - 2\frac{e}{c}\kappa_\nu p_\nu + \frac{e^2}{c^2}\kappa_\nu\kappa_\nu. \tag{15}$$

§ 4. *Relativity Quantum Mechanics.*

If we proceed to apply the method of § 2 to the systems considered in § 3, the requirements of the restricted principle of relativity will still not be completely satisfied, owing to the singular part played by the time t as a uniformising variable. To get over this difficulty we must again refer to the classical theory.

The ordinary classical theorems connecting the intensities in various directions of components of the emitted radiation with the corresponding amplitudes in the Fourier expansion of the total polarisation are valid only, if the distances moved through by the electrons during a period of the component of radiation considered are small compared with the wave length of this component, *i.e.*, if

* It has been shown by W. Wilson that the momenta defined in this way must be used in the ordinary quantum conditions $\int p\, dq = nh$ ['Roy. Soc. Proc.,' A, vol. 102, p. 478 (1923)].

the velocities of the electrons are small compared with that of light. When this condition is not satisfied, in order to determine the intensities for a given direction, say, that of the x_1 axis, one must obtain the Fourier expansion of the total polarisation in the form

$$\tfrac{1}{2}\Sigma_\alpha C_\alpha \exp \{i(\alpha\omega)(t - x_1/c)\} \tag{16}$$

where the $(\alpha\omega)$'s are constants, and are the frequencies (multiplied by 2π) of the radiation emitted in this direction, and must use these amplitudes C_α instead of the usual ones. This is readily seen to be so from the fact that the interchange of energy between the system and a field of radiation moving in the direction of the x_1 axis of frequency $(\alpha\omega)$, is governed entirely by the corresponding coefficient C_α defined by (16). The x_1 in the expression (16) refers to the point at which the charge is supposed to be concentrated. If there are several charges contributing to the total polarisation, the Fourier expansion (16) of each must be obtained separately with its respective x_1, and their corresponding amplitudes can then be added. In this case one can approximate, if the relative displacements of the charges are small, by taking the x_1 of (16) to be the x_1 of the centre of gravity of the system.

Further, if the total polarisation contains a part that increases uniformly in addition to a periodically varying part, which will occur when the whole system is charged and is moving uniformly, the non-periodic term to be added to (16) must be of the form, a constant times $(t - x_1/c)$, instead of a constant times t as in the elementary theory, in order that its contribution to the exchange of energy with the radiation field previously considered may vanish. The approximation of taking x_1 to refer to the centre of gravity of the system is not in general valid for this non-periodic term unless the velocity of the centre of gravity is small, and the theory would then reduce to the ordinary theory.

It should be noted that the amplitudes C_α determine directly the rate per unit area (I_1, say) at which energy of the radiation passes a fixed point at a distance r (a c-number) from the emitting system in the direction of the x_1 axis, by means of the formula

$$I_1 = \frac{e^2(\alpha\omega)^4}{8\pi c^3 r^2} \mid C_\alpha \mid^2; \tag{17}$$

and determine the rate of emission of energy by the system in the x_1 direction only through formulæ involving the velocity of the centre of gravity of the system, which will be found later to be ambiguous on the quantum theory. The distinction is important because the intensity I_1 is an observable quantity, while the rate of emission of energy by the system is not.

To express the theory of this § in terms of canonical variables, we observe that the only essential modification in the previous theory required is that our standard of a "uniformly increasing variable" must be changed from t to $(t - x_1/c)$. This can be effected, on both the classical and quantum theories, simply by taking $(t - x_1/c)$ to be a uniformising variable instead of t in the second of the conditions to be satisfied by the uniformising variables (§ 2). Of course this can be done only when one knows what to take for x_1, and at present the only cases in which the x_1 of expression (16) has a definite meaning are those when there is only one charged particle, and when one is able to take the x_1 of the centre of gravity as a sufficient approximation. The method of procedure in the general case is not yet known. The frequencies given by the theory with $(t - x_1/c)$ for a uniformising variable are the $(\alpha\omega)$'s of expression (16), which are the wave frequencies and not the frequencies of vibration of the system.

An example of the first of these cases in which x_1 has a definite meaning will be given in the next §, and an example of the second will now be considered. Take the system considered in the previous § in the absence of an external field, when the Hamiltonian equation is (9), and apply the canonical transformation

$$\left.\begin{array}{ll} t' = t - x_1/c & -\mathrm{W}' = -\mathrm{W} \\ x_1' = x_1 & p_1' = p_1 - \mathrm{W}/c \end{array}\right\}. \tag{18}$$

The Hamiltonian equation becomes

$$-2p_1'\mathrm{W}'/c - p_1'^2 - p_2^2 - p_3^2 = m^2c^2.$$

If we wish to consider the radiation emitted in the direction of the x_1 axis, we must take t' to be a uniformising variable, and may take for the other uniformising variables $-\mathrm{W}'$, conjugate to t', and p_1', x_1'; p_2, x_2 and p_3, x_3, together with certain J's and w's that are functions of the internal variables only.

Now consider a particular component of the emitted radiation, say that corresponding to e^{iw}. We know that w commutes with p_2, p_3 and p_1', so that

$$p_2 e^{iw} = e^{iw} p_2, \qquad p_3 e^{iw} = e^{iw} p_3$$
$$(p_1 - \mathrm{W}/c)\, e^{iw} = e^{iw}\, (p_1 - \mathrm{W}/c).$$

Hence, according to the principles of § 1, the particular c-number values possessed by p_2, p_3 and $p_1 - \mathrm{W}/c$ before the transition are equal to those they possess after the transition, so that p_2 and p_3 are unchanged by the transition, while the change in p_1 equals $1/c$ times the change in the energy W. Hence, according to the present theory, the system experiences a recoil when it emits radiation, in agreement with the light-quantum theory. Each component of the emitted radiation is associated with two momenta of the whole system as well as with two energies.

§ 5. *Theory of Compton Scattering.*

Consider a free electron subjected to plane polarised monochromatic incident radiation. The electron and incident radiation together may be considered to form a dynamical system whose emission spectrum can be determined by the methods of the preceding §§, although it is usually called not an emission spectrum but a scattered radiation.

Suppose the incident radiation to be moving in the direction of the x_1 axis with its electric vector in the direction of the x_2 axis. The electromagnetic field may then be described by the potentials

$$\kappa_1 = \kappa_3 = \kappa_4 = 0, \qquad \kappa_2 = a \cos \nu (ct - x_1) \tag{19}$$

where ν is 2π times the wave number of the incident radiation, and a determines the intensity of the incident radiation I_0 through the formula

$$I_0 = c\, a^2\nu^2/8\pi. \tag{20}$$

Since ν and I_0 can be measured physically they are c-numbers, and therefore so also is a. We shall suppose a to be small, and shall neglect second order effects. The Hamiltonian equation is, if one puts $-e$ for e in (13) and uses the values for the κ's given by (19),

$$m^2c^2 = W^2/c^2 - p_1{}^2 - \{p_2 + a' \cos \nu (ct - x_1)\}^2 - p_3{}^2 \tag{21}$$

where

$$a' = ea/c \tag{22}$$

and is a c-number. Since there are no internal co-ordinates, m is now a c-number, being the rest-mass of an electron.

We shall determine the frequency and intensity of the radiation emitted in the direction defined by the direction cosines l_1, l_2, l_3 (c-numbers). This requires that $t' = t - (l_1x_1 + l_2x_2 + l_3x_3)/c$ shall be a uniformising variable. Apply the linear canonical transformation

$$\left.\begin{aligned} x_1{}' &= ct - x_1 & p_1 &= -p_1{}' + l_1 W'/c \\ x_2{}' &= x_2 & p_2 &= p_2{}' + l_2 W'/c \\ x_3{}' &= x_3 & p_3 &= p_3{}' + l_3 W'/c \\ t' &= t - (l_1x_1 + l_2x_2 + l_3x_3)/c & -W &= -W' + cp_1{}' \end{aligned}\right\} \tag{23}$$

which gives

$$\left.\begin{aligned} (1 - l_1)\, p_1' &= - p_1 + l_1 \mathrm{W}/c \\ (1 - l_1)\, p_2' &= l_2 p_1 + (1 - l_1)\, p_2 - l_2 \mathrm{W}/c \\ (1 - l_1)\, p_3' &= l_3 p_1 + (1 - l_1)\, p_3 - l_3 \mathrm{W}/c \\ (1 - l_1)\, \mathrm{W}' &= \mathrm{W} - c p_1 \end{aligned}\right\} . \tag{24}$$

The Hamiltonian equation (21) becomes, if one neglects a^2,

$$\begin{aligned} m^2c^2 = (- \mathrm{W}' + c p_1')^2/c^2 - (- p_1' + l_1 \mathrm{W}'/c)^2 \\ - (p_2' + l_2\, \mathrm{W}'/c + a' \cos \nu x_1')^2 - (p_3' + l_3\, \mathrm{W}'/c)^2 \\ = - 2\mathrm{W}'/c\ .\ \mathrm{A} - \mathrm{B} \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathrm{A} &= (1 - l_1)\, p_1' + l_2 p_2' + l_3 p_3' + l_2\, a' \cos \nu x_1' \\ &= l_1 p_1 + l_2 p_2 + l_3 p_3 - \mathrm{W}/c + l_2 a' \cos \nu x_1' \end{aligned}$$

and

$$\mathrm{B} = p_2'^2 + p_3'^2 + 2a' p_2' \cos \nu x_1'.$$

Equation (25) takes the standard form

$$\mathrm{H} - \mathrm{W}' = 0 \tag{26}$$

where

$$\mathrm{H} = - \tfrac{1}{2} c\, (m^2c^2 + \mathrm{B})\, \mathrm{A}^{-1}. \tag{27}$$

Since W' commutes with A, we could equally well have written (25) in the form

$$m^2c^2 = - 2\mathrm{A}\mathrm{W}'/c - \mathrm{B}$$

which would have given equation (26) with

$$\mathrm{H} = - \tfrac{1}{2} c\, \mathrm{A}^{-1}\, (m^2c^2 + \mathrm{B}). \tag{27'}$$

This does not agree with (27) since A does not commute with B. More generally we could easily obtain the Hamiltonian

$$\mathrm{H} = - \tfrac{1}{2} c f_1\, (m^2c^2 + \mathrm{B}) f_2, \tag{28}$$

where f_1 and f_2 are any two functions of the single variable A such that $f_1 f_2 = \mathrm{A}^{-1}$. We are thus led to an inconsistency, as is always liable to happen when one is dealing with the Hamiltonian equation, or any other equation that does not correspond to an identity on the classical theory.

We can get over the difficulty in the present case by showing that all Hamiltonians of the type (28) give the same values for the frequency and intensity of the emitted radiation. Let

$$\mathrm{H}^* = - \tfrac{1}{2} c f_1^*\, (m^2c^2 + \mathrm{B}) f_2^*$$

be another such Hamiltonian, *i.e.*, f_1^* and f_2^* are functions of the single variable

* The notation α/β is used only when α and β commute, so there is no ambiguity.

A such that $f_1^* f_2^* = A^{-1}$, and put $f_1^* = bf_1$, so that b must be a function of the single variable A, and must commute with the f's. We must then have $f_2^* = f_2 b^{-1}$, so that H* must be connected with the H of equation (28) by the relation

$$H^* = b\,H\,b^{-1}$$

If J_r, w_r are the uniformising variables when the Hamiltonian is H, then it is easily seen that $J_r^* = bJ_r b^{-1}$, $w_r^* = bw_r b^{-1}$, which are connected with the J_r, w_r by a contact transformation, are the uniformising variables when the Hamiltonian is H*, and that H* is the same function of the J_r^* that H is of the J_r. It follows that the frequencies are the same with either Hamiltonian. Further, if X is any function of the variables of the system, then $b\,X\,b^{-1}$ must be the same function of the J_r^*, W_r^* that X is of the J_r, w_r. Now take X to be the polarisation in any direction perpendicular to the direction of emission (these being the only components of the polarisation that matter), so that

$$X = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$

where the λ's are c-numbers satisfying $\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 = 0$. We find

$$[A, X] = \lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 = 0,$$

so that X commutes with A, and therefore with b. We now have that X is the same function of the J_r^*, w_r^* as it is of the J_r, w_r, and hence its Fourier amplitudes are the same for either Hamiltonian.

Having thus established that all Hamiltonians of the type (28) lead to the same results, we may proceed, using the Hamiltonian (27), which is the most convenient one. We see at once that p_2' and p_3' commute with H, and they are therefore constants. The action and angle variables are easily verified to be*

$$J = \frac{1}{\nu(1-l_1)}\{(1-l_1)p_1' + l_2 p_2' + l_3 p_3' + l_2 a' \cos \nu x_1'\} \times \frac{m^2c^2 + p_2'^2 + p_3'^2}{m^2c^2 + p_2'^2 + p_3'^2 + 2a'p_2' \cos \nu x_1'}$$

$$= \frac{1}{\nu(1-l_1)} A \frac{m^2c^2 + B_0}{m^2c^2 + B} \qquad (29)$$

where

$$B_0 = p_2'^2 + p_3'^2$$

and

$$w = \nu x_1' + \frac{2a'p_2' \sin \nu x_1'}{m^2c^2 + B_0}, \qquad (30)$$

* *Loc. cit.*, p. 417.

since we then have

$$[w, \mathrm{J}] = \frac{1}{\nu(1 - l_1)} [w, \mathrm{A}] \frac{m^2c^2 + \mathrm{B}_0}{m^2c^2 + \mathrm{B}}$$

$$= \frac{1}{\nu}\left\{\nu + \frac{2a'p_2' [\sin \nu x_1', p_1']}{m^2c^2 + \mathrm{B}_0}\right\} \frac{m^2c^2 + \mathrm{B}_0}{m^2c^2 + \mathrm{B}} = 1,$$

as

$$[\sin \nu x_1', p_1'] = \nu \cos \nu x_1'.$$

From (27) and (29) we have

$$\mathrm{H} = -\tfrac{1}{2}c \frac{m^2c^2 + p_2'^2 + p_3'^2}{\nu(1 - l_1)\mathrm{J}}. \tag{31}$$

Since p_2' and p_3' commute with J and w, we may take them to be uniformising variables, as we then have H a function of p_2', p_3' and J only. We do not require to determine the uniformising variables conjugate to p_2' and p_3'.

There is only one component of radiation emitted, namely, that corresponding to e^{iw}. Since p_2' and p_3' commute with e^{iw}, it follows from § 1 that their c-number values remain unchanged during a transition, while the value of J is reduced by h. Thus, if we use the symbol Δ to denote the increase in the c-number value of any constant of integration during a transition, we have

$$\Delta p_2' = 0, \quad \Delta p_3' = 0, \quad \Delta \mathrm{J} = -h,$$

while if ν' is 2π times the wave number of the emitted radiation, we have from Bohr's frequency condition

$$\Delta \mathrm{H} = -ch\nu' = \Delta \mathrm{W}'.$$

If we neglect small quantities proportional to a or a', the equation $\Delta\mathrm{J} = -h$ gives, from (29),

$$\Delta \mathrm{A} = -h\nu(1 - l_1)$$

or

$$\Delta p_1' = -h\nu.$$

We now find, using the transformation equations (23),

$$\left.\begin{aligned} \Delta p_1 &= h\nu - l_1 h\nu' \\ \Delta p_2 &= -l_2 h\nu' \\ \Delta p_3 &= -l_3 h\nu' \\ \Delta \mathrm{W}/c &= h\nu - h\nu' \end{aligned}\right\}. \tag{32}$$

If one neglects a, then p_1, p_2, p_3 and W are the ordinary momenta and kinetic energy of the electron, and equations (32) are then the equations that express the conservation of momentum and energy on Compton's light-quantum theory of scattering.* The present theory thus gives the same values for the frequency of the scattered radiation and the recoil momentum of the electron as the light-quantum theory.

§ 6. *Intensity of the Scattered Radiation.*

To obtain the intensity of the emitted radiation, we must determine the amplitudes of vibration in two mutually perpendicular directions that are both perpendicular to the direction of emission (l_1, l_2, l_3). We may take the direction cosines of these two directions to be

$$l_3, \quad -\frac{l_2 l_3}{1-l_1}, \quad \frac{l_2^2}{1-l_1} - l_1 \quad \text{and} \quad l_2, \quad \frac{l_3^2}{1-l_1} - l_1, \quad -\frac{l_2 l_3}{1-l_1},$$

which are easily verified to satisfy all the necessary conditions, and put

$$\left.\begin{aligned} \mathrm{X} &= l_3 x_1 - \frac{l_2 l_3}{1-l_1} x_2 + \left(\frac{l_2^2}{1-l_1} - l_1\right) x_3 \\ \mathrm{Y} &= l_2 x_1 + \left(\frac{l_3^2}{1-l_1} - l_1\right) x_2 - \frac{l_2 l_3}{1-l_1} x_3 \end{aligned}\right\}. \tag{33}$$

We have from (27)

$$[x_1', \mathrm{H}] = \tfrac{1}{2} c\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-1}[x_1', \mathrm{A}]\,\mathrm{A}^{-1} = \tfrac{1}{2} c\,(1-l_1)\,(m^2c^2 + \mathrm{B})\ \mathrm{A}^{-2}$$

$$\begin{aligned} [\sin \nu x_1', \mathrm{H}] &= \tfrac{1}{2} c\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-1}[\sin\ \nu x_1', \mathrm{A}]\,\mathrm{A}^{-1} \\ &= \tfrac{1}{2} c\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-1}\nu\,(1-l_1)\ \cos\ \nu x_1' \,.\, \mathrm{A}^{-1} \\ &= -\nu(1-l_1)\ \mathrm{H}\ \cos\ \nu x_1' \,.\, \mathrm{A}^{-1} \end{aligned}$$

$$\begin{aligned} [x_3', \mathrm{H}] &= -\tfrac{1}{2} c\,[x_3', \mathrm{B}]\,\mathrm{A}^{-1} + \tfrac{1}{2} c\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-1}[x_3', \mathrm{A}]\,\mathrm{A}^{-1} \\ &= -\tfrac{1}{2} c \,.\, 2p_3'\mathrm{A}^{-1} + \tfrac{1}{2} c l_3\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-2} \\ &= \tfrac{1}{2} c\,(l_3 + cp_3'/\mathrm{H})\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-2} \\ &= (l_3 + cp_3'/\mathrm{H})/(1-l_1) \,.\, [x_1'\ \mathrm{H}], \end{aligned}$$

so that

$$[(1-l_1)\,x_3' - (l_3 + cp_3'/\mathrm{H})\,x_1', \mathrm{H}] = 0,$$

or

$$(1-l_1)\,x_3' - (l_3 + cp_3'/\mathrm{H})\,x_1' = \text{const.} \tag{34}$$

Also

$$\begin{aligned} [x_2', \mathrm{H}] &= -\tfrac{1}{2} c\,[x_2', \mathrm{B}]\,\mathrm{A}^{-1} + \tfrac{1}{2} c\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-1}[x_2', \mathrm{A}]\,\mathrm{A}^{-1} \\ &= -\tfrac{1}{2} c\,(2p_2' + 2a'\ \cos\ \nu x_1')\,\mathrm{A}^{-1} + \tfrac{1}{2} c l_2\,(m^2c^2 + \mathrm{B})\,\mathrm{A}^{-2} \\ &= -\,a'c\cos\nu x_1'.\quad \mathrm{A}^{-1} + \tfrac{1}{2} c\,(l_2 + cp_2'/\mathrm{H})\,(m^2c^2 + \mathrm{B})\ \mathrm{A}^{-2} \\ &= a'c/\nu\,(1-l_1)\,\mathrm{H} \,.\, [\sin\ \nu x_1', \mathrm{H}] + (l_2 + cp_2'/\mathrm{H})/(1-l_1) \,.\, [x_1', \mathrm{H}], \end{aligned}$$

so that

$$(1-l_1)x_2' - a'c/\nu\mathrm{H} \,.\, \sin\ \nu x_1' - (l_2 + cp_2'/\mathrm{H})x_1' = \text{const.} \tag{35}$$

* Compton, 'Phys. Rev.,' vol. 21, p. 483 (1923).

From equations (23) we find

$$x_2 = x_2', \quad x_3 = x_3', \quad (1 - l_1)\, x_1 = ct' - x_1' + l_2 x_2' + l_3 x_3',$$

so that the first of equations (33) may be written

$$\begin{aligned}(1 - l_1)\, \mathrm{X} &= l_3\,(ct' - x_1' + l_2 x_2' + l_3 x_3') - l_2 l_3 x_2' + (l_2^2 - l_1 + l_1^2)\, x_3' \\ &= l_3 ct' - l_3 x_1' + (1 - l_1) x_3' \\ &= l_3 ct' + cp_3'/\mathrm{H} \,.\, x_1' + \text{const.}\end{aligned}$$

with the help of (34), and similarly the second of equations (33) may be written

$$\begin{aligned}(1 - l_1)\, \mathrm{Y} &= l_2\,(ct' - x_1' + l_2 x_2' + l_3 x_3') + (l_3^2 - l_1 + l_1^2)\, x_2' - l_2 l_3 x_3' \\ &= l_2 ct' - l_2 x_1' + (1 - l_1)\, x_2' \\ &= l_2 ct' + cp_2'/\mathrm{H} \,.\, x_1' + a'c/\nu\mathrm{H} \,.\, \sin \nu x_1' + \text{const.}\end{aligned}$$

with the help of (35).

We are interested only in the periodic parts of X and Y, and may omit the constant parts and the parts that increase uniformly with respect to t' or w. To the first order in a equation (30) for w may be written

$$\nu x_1' = w - 2a' p_2'/(m^2c^2 + \mathrm{B}_0) \,.\, \sin w,$$

and we now find for the periodic parts of x and y, with the help of (31),

$$\left.\begin{aligned}\mathrm{X} &= -\frac{cp_3'}{(1 - l_1)\,\mathrm{H}} \frac{2a'p_2'}{\nu\,(m^2c^2 + \mathrm{B}_0)} \sin w = \frac{4a'p_2'p_3'\mathrm{J}}{(m^2c^2 + \mathrm{B}_0)^2} \sin w \\ \mathrm{Y} &= \left\{-\frac{cp_2'}{(1 - l_1)\,\mathrm{H}} \frac{2a'p_2'}{\nu\,(m^2c^2 + \mathrm{B}_0)} + \frac{a'c}{(1 - l_1)\,\nu\mathrm{H}}\right\} \sin w \\ &= -\frac{2a'\,(m^2c^2 - p_2'^2 + p_3'^2)\,\mathrm{J}}{(m^2c^2 + \mathrm{B}_0)^2} \sin w\end{aligned}\right\} \quad (36)$$

These equations may be written

$$\begin{aligned}\mathrm{X} &= -2ia' \left\{\frac{p_2'p_3'\mathrm{J}}{(m^2c^2 + \mathrm{B}_0)^2} e^{iw} - e^{-iw} \frac{p_2'p_3'\,(\mathrm{J} - h)}{(m^2c^2 + \mathrm{B}_0)^2}\right\} \\ \mathrm{Y} &= ia' \left\{\frac{(m^2c^2 - p_2'^2 + p_3'^2)\,\mathrm{J}}{(m^2c^2 + \mathrm{B}_0)^2} e^{iw} - e^{-iw} \frac{(m^2c^2 - p_2'^2 + p_3'^2)\,(\mathrm{J} - h)}{(m^2c^2 + \mathrm{B}_0)^2}\right\}.\end{aligned}$$

The coefficients in front of e^{iw} and behind e^{-iw} in the expansion of X or Y are not conjugate imaginaries, owing to the fact that J and w are not real. All the same, their product must still be a quarter of the square of the amplitude of vibration, expressed as a function of the initial value of the action variable. We thus obtain for the sum of the squares of the amplitudes of X and Y the value

$$\begin{aligned}\mathrm{C}^2 &= 4\, a'^2 \{4\, p_2'^2 p_3'^2 + (m^2c^2 - p_2'^2 + p_3'^2)^2\}\, \mathrm{J}\,(\mathrm{J} - h)/(m^2c^2 + \mathrm{B}_0)^4. \\ &= \frac{4a'^2\,\mathrm{J}\,(\mathrm{J} - h)}{(m^2c^2 + p_2'^2 + p_3'^2)^2} \left\{1 - \frac{4m^2c^2p_2'^2}{(m^2c^2 + p_2'^2 + p_3'^2)^2}\right\}. \qquad (37)\end{aligned}$$

If the electron is initially at rest, except for the small oscillations caused by the incident radiation, we must substitute for p_2', p_3' and J their values determined by the relations

$$p_1 = p_2 = p_3 = 0, \qquad W = mc^2$$

which give, from (24)

$$p_1' = l_1 mc/(1 - l_1) \qquad p_2' = - l_2 mc/(1 - l_1) \qquad p_3' = - l_3 mc/(1 - l_1)$$

$$m^2c^2 + B_0 = m^2c^2 + p_2'^2 + p_3'^2 = 2m^2c^2/(1 - l_1)$$

and

$$J = A/\nu(1 - l_1) = - mc/\nu(1 - l_1)$$

with neglect of a. The value of C^2 given by (37) now reduces to

$$C^2 = \frac{a'^2}{m^2c^2\nu^2}(1 - l_2^2)\left(1 + \frac{h\nu(1 - l_1)}{mc}\right) = \frac{a'^2}{m^2c^2\nu^2}(1 - l_2^2)\frac{\nu}{\nu'}$$

if we use the Compton relation connecting ν' with ν, namely,

$$\frac{1}{\nu'} = \frac{1}{\nu} + \frac{h(1 - l_1)}{mc}.$$

The intensity of the emitted radiation at a distance r from the emitting electron is now given by equation (17) with $c\nu'$ substituted for $(\alpha\omega)$, *i.e.*,

$$I = \frac{e^2c^4\nu'^4}{8\pi c^3 r^2}\frac{a'^2}{m^2c^2\nu^2}\frac{\nu}{\nu'}(1 - l_2^2) = \frac{e^4 I_0}{m^2c^4r^2}\frac{\nu'^3}{\nu^3}(1 - l_2^2) \tag{38}$$

with the help of (20) and (22). This is just $(\nu'/\nu)^3$ times its value according to the classical theory.

If the incident radiation is unpolarised, one must average (38) for all directions of polarisation of the incident radiation, and the result that the actual intensity is $(\nu'/\nu)^3$ times its classical value still holds. This result is not very different from Compton's formula* for the intensity of the scattered radiation. In particular, they agree when the angle of scattering is 0 or 180°.

§ 7. *Comparison with Experiment.*

The result obtained in the preceding § that the intensity of the radiation scattered by a free electron in any direction is $(\nu'/\nu)^3$ times its value, according to the classical theory, where ν'/ν is the ratio of the wave number of the radiation scattered in that direction to the wave number of the incident radiation, admits of comparison with experiment. This is the first physical result obtained from the new mechanics that had not been previously known.†

* Compton, *loc. cit.*, equation (27).

† *Note added, May,* 1926.—This result for unpolarised incident radiation has recently been obtained independently by Breit from correspondence principle arguments ('Phys. Rev.,' vol. 27, p. 362, 1926).

The quantum formula for the intensity at distance r of the radiation scattered by N electrons with plane polarised incident radiation of intensity I_0 is

$$I(\theta, \phi) = I_0 \frac{Ne^4}{r^2m^2c^4} \frac{\sin^2\phi}{\{1 + \alpha(1 - \cos\theta)\}^3}, \tag{39}$$

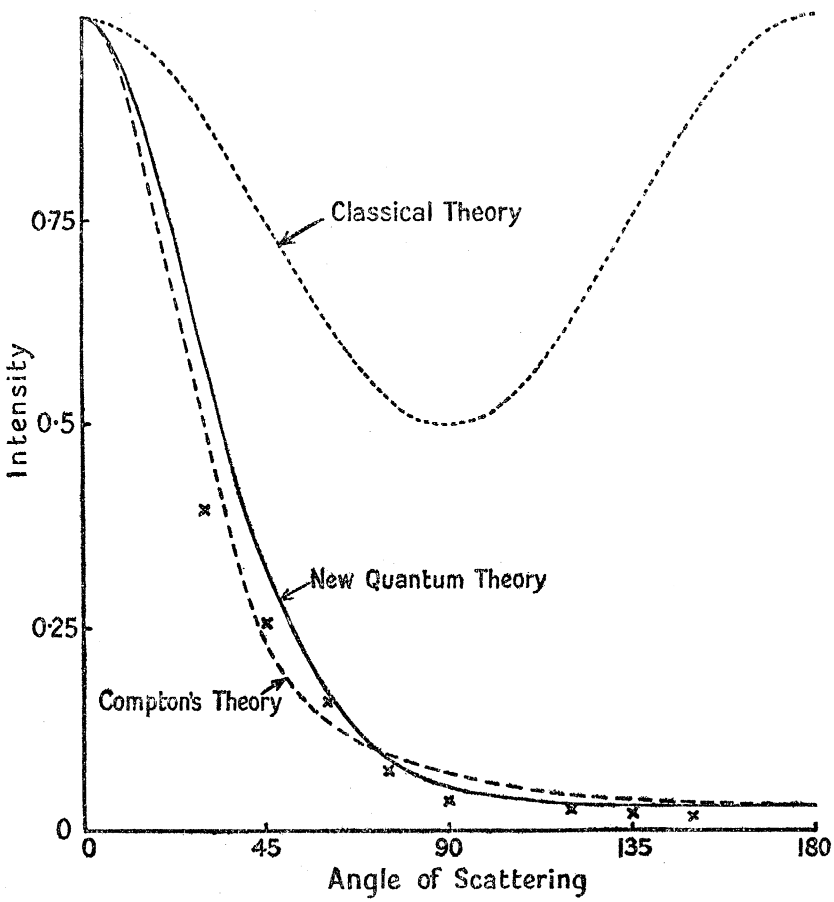
where

$$\alpha = h\nu/mc$$

and θ is the angle of scattering and ϕ the angle between the direction of the scattered radiation and the direction of the electric vector of the incident radiation. For unpolarised incident radiation the formula is

$$I(\theta) = I_0 \frac{Ne^4}{2r^2m^2c^4} \frac{1+\cos^2\theta}{\{1 + \alpha(1 - \cos\theta)\}^3}. \tag{40}$$

The full curve in the figure shows the variation of the intensity of the scattered radiation with the angle of scattering according to formula (40) for unpolarised incident radiation of wave length 0·022 Å, which makes $\alpha = 1 \cdot 1$. The lower



broken curve is the result given by Compton's theory,* and the upper broken curve is given by the classical theory. The crosses indicate experimental values obtained by Compton, which have been taken from Compton's paper.† It will be observed that the experimental values are all less than the values given by the present theory, in roughly the same ratio (75 per cent.), which shows that the theory gives the correct law of variation of intensity with angle, and suggests that in absolute magnitude Compton's values are 25 per cent. too small.

One may easily obtain a formula for the total energy removed from the primary beam by scattering, by integrating I $(\theta)\ \nu/\nu'$ over all solid angles. The result is

$$I_0 \frac{2\pi N e^4}{m^2 c^4} \frac{1+\alpha}{\alpha^2} \left\{ \frac{2(1+\alpha)}{1+2\alpha} - \frac{1}{\alpha} \log (1 + 2\alpha) \right\}$$

which for ordinary values of α lies very close to Compton's expression

$$I_0 \frac{8\pi}{3} \frac{N e^4}{m^2 c^4} \frac{1}{1+2\alpha},$$

(*e.g.*, for $\alpha = 1$ our formula gives a result 5·7 per cent. greater than Compton's), and is in very good agreement with experiment.

According to the present theory the state of polarisation of the scattered radiation is the same as on the classical theory, since the intensity of either polarised component of the scattered radiation in any direction is $(\nu'/\nu)^3$ times its classical value. The radiation scattered through 90° is thus plane polarised for unpolarised incident radiation. This result might have been expected from the correspondence principle, since it holds on the classical theory for an electron moving with either the initial velocity (*i.e.*, zero) or the final velocity of the quantum process. It does not hold for an electron recoiling with that velocity that gives the correct frequency distribution when the electron is scattering according to the classical theory, and for this reason previous theories have predicted a shift from 90° for the angle of scattering which gives plane polarisation.‡ Experiments have been performed by Jauncey and Stauss to settle this question.§ They found no shift with incident radiation of 0·54 Å, and a shift of $2\frac{1}{2}°$, less than half the value they expected, with incident radiation of 0·25 Å, these results are slightly in favour of the present theory which requires no shift. Great accuracy was not attainable owing to the difficulties caused by stray radiation.

* Compton, *loc. cit.*, equation (27). Other formulæ have been obtained by Jauncey, 'Phys. Rev.,' vol. 22, p. 233 (1923).

† Compton, *loc. cit.*, fig. (7).

‡ See Jauncey, 'Phys. Rev.,' vol. 23, p. 313 (1924).

§ Jauncey and Stauss, 'Phys. Rev.,' vol. 23, p. 762 (1924).