Dynamical Laws and Statistical Laws

Conscientiousness and truth are as necessary in research in pure science as in practical life. The experimenter must not be blinded by the first results of a new intellectual discovery and must not neglect to prove conscientiously and thoroughly the results obtained in his researches. He must keep his mind fixed firmly on his original starting-point and the methods he employs. It may happen in a day that his hardly-won position is assailed and made untenable against severe criticism. Therefore, the experimenter cannot afford to close his eyes to a new discovery, obtained from another point of view, which will not fit in with his own ideas, nor must he treat it as unimportant, if not incorrect.

Such unforeseen and unexpected discoveries occur in all sciences, particularly when they are permeated with the spirit of youth. For every science, not even excluding mathematics, is to some extent the result of observation, whether the subject be natural or intellectual. The chief problem in every science is that of endeavouring to arrange and collate the numerous individual observations and details which present themselves, in order that they may become part of one comprehensive picture.

Now, considering the different subjects comprised in the various branches of science, their laws are by no means so diverse in their natures as might appear from a glance at the marked contrasts presented, for example, by questions in history and physics. At least it would be quite incorrect to look for a fundamental difference in the fact that in the domain of natural science a law must everywhere, without exception, be absolute, and the sequence of phenomena certain. In intellectual work, however, the pursuit of causal relations leads, here

and there, always through something arbitrary and casual. On the one hand, every scientific thought involves the necessity of assuming a fixed absolute law raised above arbitrariness and chance to the highest level of the human intellect. On the other hand, even physics, the most exact of natural sciences, has frequently to deal with phenomena which cannot for the present be connected by any law, and which therefore may be considered accidental in a certain sense of the word.

Let us consider for a moment, as a special example, the behaviour of radio-active atoms according to the accepted disruption hypothesis of Rutherford and Soddy. How is it that a definite Uranium atom, after having remained completely unaltered and passive for untold millions of years, suddenly, in an immeasurably short space of time, without any determinable cause, explodes with a violence, compared with which our most powerful explosives are like toy pistols? It sends off fragments of itself with velocities of thousands of kilometres per second, and at the same time emits electro-magnetic rays of greater intensity than the hardest Röntgen rays, while another atom in its immediate neighbourhood, and to all appearances exactly similar, remains in a passive state for still more millions of years until finally it meets the same fate. In fact, all attempts to affect the course of radio-active phenomena by external means, such as raising or lowering of temperature, have ended in complete failure. It appears, therefore, at present hopeless even to guess at dynamical laws which would account for this. Yet the so-called theory of atomic disruption is of the greatest importance to physical research. It has co-ordinated, from the first, an almost embarrassing number of isolated facts, and has yielded many new results, some of which have been verified experimentally in a wonderful way, and others have stimulated new and important researches and discoveries.

How is this possible? How can practical laws be derived by considering phenomena the cause of which has, provisionally, to be left completely unexplained? Like the social sciences, physics has learnt to appreciate the great importance of a method completely different from the purely causal, and has applied it since the middle of last century with continually increasing success. This is the statistical method, and the newest advances in theoretical physics have been bound up with its development. Instead of seeking, without tangible results, the dynamical laws at present completely unknown to us, which govern a solitary occurrence, observations of a large number of isolated occurrences of a definite kind are collected and an average or mean value obtained. For the calculation of these mean values, certain empirical rules are available, according to the special circumstances of the case. These rules permit the prediction of future occurrences, not with absolute certainty, but with a probability which is often practically equivalent to certainty. This will not be true in all details, but only on the average, and that is usually what is wanted in applications.

Though a method which is fundamentally an expedient appears unsuited and unsympathetic to the scientific needs of many workers, who desire principally an elucidation of causal relations, yet it has become absolutely indispensable in practical physics. A renunciation of it would involve the abandonment of the most important of the more recent advances of physical science. It must also be borne in mind that physics, in the exact sense, does not deal with quantities that are absolutely determined; for every number obtained by physical measurements is liable to a certain possible error. Anyone who would only admit actual, definite numbers and not at the same time a possible error, would have to abandon the use of measurements and consequently all inductive knowledge.

It is sufficiently evident from the above that, in order to understand the characteristics of any science, it is of the utmost importance to differentiate carefully and fundamentally between the two classes of laws: the *dynamical*, strictly causal; and the solely *statistical*. I wish to compare and contrast these laws.

We will consider a few observations from everyday life. Let us take two open vertical glass tubes, connect the lower ends with rubber tubing, and pour into one of the tubes a quantity of a heavy liquid, such as mercury. The liquid will flow through the rubber tubing into the second tube, until the level of the surfaces in the two tubes is the same. This condition of equilibrium always returns after any disturbance. If, for example, one tube is suddenly raised, so that the mercury is for an instant raised with it, and consequently is at a higher level in that tube, it will immediately fall again until the surfaces are at the same height again in both tubes. This is the well-known principle on which every syphon action is based.

Let us take another example. We take a piece of iron, heated to a high temperature, and throw it into a vessel of cold water. The heat of the iron will be communicated to the water until complete equality of temperature is obtained. This is the so-called thermal equilibrium which will obtain after every disturbance.

There exists a certain analogy between the two examples. In each case, a certain difference brings about the variation, in the one case a difference of level and in the other a difference of temperature, and equilibrium is restored when the difference vanishes. Temperature is, therefore, sometimes referred to as level of heat. It can be said that, in the first case, the energy of gravitation, in the second case the energy of heat, flows from the higher level to the lower until the levels are the same.

It is no wonder that this analogy of a directing of energy has been explained as the action of a great general "principle of chance." This directing, though with the best intentions, has led to hasty generalizations. The principle makes each change in Nature an exchange of energy, and consider the different forms of energy as independent and of equal value. To each form of energy corresponds a factor of intensity, to gravitation height, to heat temperature, and the difference of these factors will determine the workings of chance. The confidence with which the general validity of this theorem was proclaimed is due to its simplicity and it was inevitable that it should appear early in popular expositions and elementary text-books.

Actually, the analogy between the two examples is only superficial, and the laws governing them are very widely separated. For, as all our experiences permit us confidently to assert, the first example obeys a dynamical law, the second a statistical one. Or, in other words, that liquid flows from a higher to a lower level is necessary, but that heat flows from a place of higher temperature to one of lower temperature is only probable.

It must be understood that such an assertion, which appears at first sight strange and almost paradoxical, requires to be supported by an enormous number of examples. I will endeavour to outline the most important of these and at the same time make clear the difference between dynamical and statistical laws. In the first place, that it is *necessary* for the heavy liquid to sink can easily be proved to be a consequence of the principle of conservation of energy. For if the liquid at the higher level rose to a level still higher without any external agency, and the liquid at the lower level sank still further, energy would be created out of nothing, which is contrary to the principle. The second case is somewhat different. Heat could very well flow from the cold water to the hot iron without violating the principle of conservation of energy; for, since heat is itself a form of energy, this principle only requires that the quantity of heat given up by the water is equal to that absorbed by the iron.

But the two operations show certain characteristic differences to the unbiased observer. The falling liquid moves faster the further it sinks. When the levels of the liquid are the same, the liquid does not come to rest, but moves beyond the equilibrium position on account of its inertia, so that the liquid originally at the higher level is at a lower level. Now, the velocity will decrease and the liquid will come to rest gradually, and subsequently the same process is repeated in the reverse direction. If all loss of kinetic energy at the air boundaries and that due to friction at the walls of the tube could be eliminated, the liquid would oscillate backwards and forwards indefinitely about its position of equilibrium. Such a process is, therefore, called reversible.

It is quite otherwise with heat. The smaller the difference of temperature between the iron and the water, the slower is the transmission of heat from the one to the other, and calculation shows that an infinitely long time elapses before equality of temperature is attained. In other words, there is always a small difference of temperature, however much time has elapsed.

There can be no talk of oscillation of heat between the two bodies: the flow of heat is always in one direction, and therefore represents an irreversible process.

In all physical science there is no more fundamental difference than that between reversible and irreversible processes. The former include gravitation, mechanical and electrical oscillations, acoustic and electro-magnetic waves. They can all be grouped under one single dynamical law—the principle of least action—which embraces the principle of the conservation of energy. Irreversible processes include conduction of heat and electricity, friction, diffusion and all chemical reactions, in so far as they take place with noticeable velocity. To cover these, R. Clausius derived his second law of thermo-dynamics, so exceptionally useful in physics and chemistry. The significance of this law is that it ascribes a direction to each irreversible process. But it was L. Boltzmann who, by the introduction of the atomic theory, explained the meaning of the second law and at the same time all irreversible processes, the peculiarities of which had presented insuperable difficulties of explanation by means of general dynamics.

According to the atomic theory, the heat energy of a body is simply the sum total of the extremely small, rapid, unregulated movements of its individual molecules. The temperature corresponds to the mean kinetic energy of the molecules, and the transmission of heat from a hot body to a cold body depends upon the fact that the kinetic energies of the molecules are meaned on account of the frequent collisions of the bodies. From this it must not be supposed that when two individual molecules collide, the one with the greater kinetic energy is slowed up and the other accelerated, for if, for example, a rapidly moving molecule of one system is struck obliquely by a slower moving molecule, its velocity must be still further increased, while that of the slower molecule is still further diminished. But, in general, unless the circumstances are very exceptional, the kinetic energies must mix to a certain extent, and this corresponds to an equalizing of the temperatures of the two bodies. All results deduced in this manner agree with observation, particularly in the case of gases.

However much discussed and however promising this atomic theory might appear, it was, until recently, regarded merely as a brilliant hypothesis, since it appeared to many far-sighted workers too risky to take the enormous step from the visible and directly controllable to the invisible sphere, from the macrocosm to the microcosm. In order that he should not imperil the acceptance of his observations and calculations, Boltzmann himself did not over-emphasize them. He laid stress on the view that the atomic hypothesis was a mere representation of what took place. To-day we may go further towards comparing the reality with the picture, in so far as it has any meaning at all, from the point of view of the philosopher. For to-day we have a series of experiments which invest the atomic hypothesis with the same degree of certainty as is possessed by the mechanical theory of sound, or the electro-magnetic theory of light and heat radiations.

According to the theory of chance, inadequately outlined above, the condition of a stationary fluid of uniform temperature must be absolutely invariable; for if no difference of intensity of any sort exists in the fluid, there can be no cause which will bring about any variation. The state of a fluid can be made visible by introducing into a transparent liquid, such as water, a number of minute particles or drops of another liquid, such as gummastic or gamboge. I do not think that anyone who has observed such a preparation through a properly illuminated microscope, can ever forget his first view of the play presented to him. It is a glance into a new world. Instead of the complete tranquillity he expected, he sees an extraordinarily lively, gay dance of the small floating particles, in which the smallest behave in the most erratic manner: no trace of any friction in the fluid can be seen; if a particle once comes to rest, another starts the game. One is involuntarily reminded of the frenzied activity of an ant-hill which has been disturbed. But whereas the angry insects gradually calm down and lose their activity towards dusk, the particles under the microscope never show the least signs of fatigue, while the temperature of the liquid remains unaltered—an actual case of perpetual motion, in the most literal sense of this much-used expression.

The phenomenon described was discovered in the year 1827 by Brown, the English botanist, but it had been deduced by the French physicist Gouy, twenty-five years earlier, from the movements of molecules in a heated fluid. These molecules, themselves invisible, continually collide with particles floating around them (which are visible in a microscope) and are impelled along irregular paths. The final theoretical proof of the correctness of this explanation was first given quite recently, when Einstein and Smoluchowski, obtained statistical laws governing the distribution of density, the velocities, the mean free paths, and even the rotations of the microscopic particles, and these laws were most strikingly confirmed quantitatively in all details, particularly through the experimental work of Jean Perrin.

There can be no doubt now, in the mind of the physicist who has associated himself with inductive methods, that matter is constituted of atoms, heat is movement of molecules, and conduction of heat, like all other irreversible phenomena, obeys, not dynamical, but statistical laws, namely, the laws of probability. Indeed, it is difficult to make even an approximate estimate of the probability that heat will travel in the contrary direction, i.e. from the cold water to the hot iron. If one draws one letter after another at random from a sack filled with letters, and sets them out in a row in the order in which they are drawn, there is always a *possibility* that complete words may be formed, even that they will form a poem by Goethe. Or if a hundred throws are made with a die, no one will dispute the possibility that six will turn up each time without exception, since the result of each throw is independent of the previous one. Should this occur in practice, there is no doubt that everyone would say that there was something wrong, perhaps the die was not quite symmetrical, and no rational person would deny the weight of this observation. For the probability that so exceptional an occurrence should take place under normal circumstances is extremely minute. Yet this is enormously great compared with the probability that heat will flow from a cold to a hot body. We need only consider that in the case of the die, we are dealing with six numbers, consequently with six different cases, in the case of the letters, with twenty-six, but in the case of the molecules with many millions in the smallest visible space, and moving with extremely diverse velocities. Thus from the standpoint of practical physics, there is certainly no ground to believe the possibility of a deviation from the general truth of the laws governing radiation of heat.

This is certainly not the case with the theory. For it is clear to everybody that there must be an unfathomable gulf between a probability, however small, and an absolute impossibility. This can be demonstrated in particular circumstances. One need only throw the die sufficiently often in order, with greater probability, to expect a hundred consecutive sixes, and one need only persevere sufficiently long at the letter game to obtain a Faust monologue. Still, it is as well that we do not depend solely on these methods, for neither the age of a man, nor probably that of mankind, would be long enough.

Whatever the application to physics involves, it is necessary to consider very seriously such infinitesimal probabilities under certain conditions. If a powder magazine were to explode without any determinable cause, the occurrence would not be ignored. The so-called self-ignition is to be regarded as caused by a very improbable accumulation of dangerous impacts of chemically reacting molecules; the laws governing these molecules can only be arrived at statistically. It is obvious that in an exact science such words as *certain* and *sure* must be used with great caution, and the importance of the laws of observation must be very moderately assessed. Thus, when considering the laws of physics, or, indeed, any observed law, either dynamical or statistical, we are compelled by theory and experiment alike to make a fundamental difference between necessity and probability. This duality, which has been brought into all physical laws by the introduction of statistical methods, will appear unsatisfactory to many. Accordingly, when it appeared unsuitable, efforts were made to set it aside by denying absolute certainty and impossibility, and substituting great and small degrees of probability respectively. If there were no dynamical laws in Nature, but only statistical, the conception of absolute necessity would have no place in physics. Such a view must very

soon prove to be a mistake as dangerous as it is short-sighted, apart from the fact that all reversible processes, without exception, are governed by dynamical laws, and that we have no reasons for discarding these laws. Physics can no more do without the hypothesis of absolute laws than can any other natural science or human study, for without it the essential foundations of deductions from statistics would be removed, and it is these deductions that we are considering.

Yet one considers that the theorems of the calculus of probability are not only capable of, but also require, a strict exposition and rigid proof, and therefore it has always particularly attracted prominent mathematicians. If the probability that a certain event is succeeded by a certain other event is $\frac{1}{2}$, then, it can be said that nothing is known of the occurrence of the second event, except that it will follow in just 50 per cent. of the cases when the first event occurred, and that this percentage is more nearly obtained the greater the number of cases that are considered. In addition, the calculus of probability furnishes an exact estimate of the deviation from the mean, which is to be expected when the number of observed cases is smaller, i.e. of the so-called dispersion. If the observations are in contradiction to the calculated magnitude of the dispersion, it may be safely concluded that an erroneous assumption was made in the premises, a so-called systematic error.

To support such far-reaching assertions, very extensive presuppositions are naturally essential, and it will be understood that in physics the exact calculation of probabilities is only possible when purely dynamical laws can be assumed to hold in the simplest occurrences, i.e. in the smallest microcosm. Should these laws contradict a single observation through our fallibility, the hypothesis of their absolute immutability furnishes a necessary firm foundation for the structure of statistics.

It appears from these remarks that the duality between statistical and dynamical laws is intimately associated with the duality between macrocosm and microcosm, and this we must regard as a fact substantiated by experiment. Whether satisfactory or not, facts cannot be created by theories, and there is no alternative but to concede their appointed places to dynamical as well as to statistical laws in the whole system of physical theories.

Thus dynamics and statistics cannot be regarded as interrelated. For, whereas a dynamical law completely satisfies the causal requirements and is therefore of a simple character, every statistical law is built up, and it cannot in any way be looked on as definitive, since it always involves the problem of reduction to its simple dynamical elements. The solution of such problems is one of the chief tasks of progressive science. This is as much the case in chemistry as in the physical theories of matter and in electricity. Meteorology may also be mentioned in this connection, for the work of V. Bjerknes provides a scheme of great magnitude to reduce all meteorological statistics to their simpler elements, namely, to physical laws. Whether the attempt leads to practical results or not, it must be made at some time, since the essence of all statistics is that while it often has the first, it never has the last, word.

As the principle of conservation of energy or the first law of thermo-dynamics occupies the first place among the dynamical laws of physics, so the second law of thermo-dynamics holds a corresponding place among the statistical laws. Although this theorem is a probability theorem, and, in consequence, one often speaks of limits to its validity, it can be expressed in an exact and generally valid form. It might be expressed somewhat as follows: All physical and chemical changes of state proceed, on the average, towards states of greater probability. Of all the states that can be assumed by a system of bodies, the most probable is that in which all the bodies have the same temperature. On this ground only is based the law that heat conduction always, *on the average*, tends towards an equalization of temperatures, and also from the higher to the lower temperature. The second law will only allow us to deduce anything with certainty from a *single* observation if we are certain beforehand that the course of the operation in question is not markedly different from the mean course deduced from a large number of operations in which the initial conditions were the same. To make sure that this condition is satisfied, it is, theoretically, sufficient to introduce the so-called hypothesis of elementary

disorder. Experimentally, the only method is to repeat the particular observation many times, or to have it done by different observers, working independently of one another. Such a repetition of a definite experiment, or the arranging of a whole series of experiments, is actually what is done in practical physics. For, in order to eliminate unavoidable errors of observation, no physicist will limit himself to the results of a single experiment.

The second law of thermo-dynamics has nothing to do with energy directly. A good example of a process which need not be accompanied by a transformation of energy is diffusion. Diffusion happens solely because a uniform mixing of two different substances is more probable than a non-uniform mixing. This can, indeed, be subordinated to the conception of energy, by introducing, for this special purpose, the idea of free energy, which permits of a convenient exposition and in many cases simplifies the representation. The method, however, is indirect in so far as free energy can only be understood from its relation to probability.

Let us, in conclusion, pause awhile after this rapid survey to consider the laws of phenomena in the intellectual sphere. To a great extent, we find quite similar relations, except that causality is completely eclipsed by probability, the microcosm by the macrocosm. Yet here in all questions extending to the highest problems of intellect and morality, the assumption of absolute determinism is a necessary basis for every scientific investigation. Care must be taken that the normal course of the phenomenon examined is not disturbed by the examination. This is equally true in natural science, but not usually emphasized on account of its being almost self-evident. When a physicist wants to take the temperature of a body, he does not use a thermometer the introduction of which would alter the temperature of the body. From this point of view the possibility of a completely objective scientific investigation into psychological phenomena only extends to the critical examination of personalities other than the observer, so long as they are independent of the observer. In so far as it is completely effaced from the mind of the investigator, it also extends to the past, but not to the present, nor to the future, which must always be attained through the present. Thought and research are themselves psychological phenomena in man, and if the object of the investigation is identical with the investigator, he must change continually as his knowledge advances.

It is, therefore, quite useless to treat exhaustively the phenomena of the future from the standpoint of determinism, and with it to wish to fix the conception of moral freedom. Self-determination is given to us by our consciousness and is not limited by any causal law, and he who considers it logically irreconcilable with absolute determinism in all spheres of philosophy, makes a great mistake of the same nature as that made by the physicist already referred to, who does not take adequate precautions to eliminate errors in his observations, or a mistake such as a physiologist would make if he examined himself in order to study the functioning of a muscle in anatomy.

Science thus fixes for itself its own inviolable boundaries. But man, with his unlimited impulses, cannot be satisfied with this limitation. He must overstep it, since he needs an answer to the most important, and constantly-repeated question of his life: What am I to do?—And a complete answer to this question is not furnished by determinism, not by causality, especially not by pure science, but only by his moral sense, by his character, by his outlook on the world. Conscientiousness and truth are the ideals that will lead him along the true path in life as in science. They will guarantee him, not necessarily brilliant results, but the highest good of humanity, namely, inward peace and true freedom.

The Principle of Least Action

As long as physical science exists, the highest goal to which it aspires is the solution of the problem of embracing all natural phenomena, observed and still to be observed, in one simple principle which will allow all past and, especially, future occurrences to be calculated. It follows from the nature of things, that this object neither has been, nor ever will be, completely attained. It is, however, possible to approach it nearer and nearer, and the history of theoretical physics shows that already an extensive series of important results can be obtained, which indicates clearly that the ideal problem is not purely Utopian, but that it is eminently practicable. Therefore, from a practical point of view, the ultimate object of research must be borne in mind.

Among the more or less general laws, the discovery of which characterize the development of physical science during the last century, the principle of Least Action is at present certainly one which, by its form and comprehensiveness, may be said to have approached most closely to the ideal aim of theoretical inquiry. Its significance, properly understood, extends, not only to mechanical processes, but also to thermal and electrodynamic problems. In all the branches of science to which it applies, it gives, not only an explanation of certain characteristics of phenomena at present encountered, but furnishes rules whereby their variations with time and space can be completely determined. It provides the answers to all questions relating to them, provided only that the necessary constants are known and the underlying external conditions appropriately chosen.

This central position attained by the principle of least action is, however, not even to-day quite undisputed; for a long time