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THE IDENTITY OF INDIVIDUALS IN A STRICT FUNCTIONAL CALCULUS OF SECOND ORDER

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In previous papers[1] we developed two functional calculi of first order based on strict implication which we called $S2^1$ and $S4^1$. In the present paper,[2] these systems will be extended to include a functional calculus of second order with the purpose of introducing the relation of identity of individuals.

Primitive symbols. $\hat{}$ {the abstraction operator, the blank space to be replaced by an appropriate variable}.

Syntactic notation. The Greek letters ρ, ρ_1, $\cdots$, ρ_n represent propositional variables. $\theta, \phi, \psi, \theta_1, \phi_1, \psi_1, \cdots, \theta_n, \phi_n, \psi_n$ represent functional variables of individuals. $\gamma, \delta, \gamma_1, \delta_1, \cdots, \gamma_n, \delta_n$ will hereafter represent variables the type of which will be specified.

Extension of the definiton of well-formed-formula. If A is a wff then $(\exists\gamma)$A is a wff where γ may be a propositional or a functional variable. If A is a wff then $\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_nA(\beta_1, \beta_2, \cdots, \beta_n)$ is a wff where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are distinct individual variables and $\beta_1, \beta_2, \cdots, \beta_n$ are individual variables not necessarily distinct.

An expression of the form $\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_n$A will be termed an abstract.

Extension of the definition of free and bound variable. An occurrence of a variable γ in a wff A is a bound occurrence if it is in a wf part of A of the form $(\exists\gamma)$B where γ may be a functional or a propositional variable as well as an individual variable. Otherwise it is a free occurrence. An occurrence of an individual variable β will be said to be free in the abstract $\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_n$A if it is distinct from $\alpha_1, \alpha_2, \cdots, \alpha_n$ and is free in A. Otherwise it is a bound occurrence.

Axiom schemata. 1.9, 1.10 and 1.11 are extended over functional variables of individuals and propositional variables.

2.1. (ρ)A $\prec$ B where A, B, and Γ are wff, ρ is a propositional variable, no free occurrence of ρ in A is in a wf part of A of the form (γ)E where γ is a variable free in Γ and B results from A by replacing all free occurrences of is a variable free in Γ and B results from A by replacing all free occurrences of ρ in A by Γ.

2.2. (θ)A $\prec$ B where θ is a functional variable, A, B and H are wff's, no

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[1] R. C. Barcan, *A functional calculus of first order based on strict implication*, this JOURNAL, vol. 11 (1946), pp. 1–16 and *The deduction theorem in a functional calculus of first order based on strict implication*, this JOURNAL, vol. 11 (1946), pp. 115–118. The reader is asked to make the following changes in the enumeration of the axioms and theorems appearing in these papers: Place "1." before the number of each theorem, e.g., axiom 1 will be referred to as 1.1 etc.

[2] Part of this paper was included in a dissertation written in partial fulfillment of the requirements for the Ph.D. degree in Philosophy at Yale University. I am indebted to Professor Frederic B. Fitch for his criticisms and suggestions.

free occurrence of θ in A is in a wf part of A of the form $(\gamma)\Gamma$ where γ is any variable free in $\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_n$H and B results from A by replacing all free occurrences of θ in A by $\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_n$H.[3]

2.3. $\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_n A(\beta_1, \beta_2, \cdots, \beta_n) \equiv B$ where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are distinct individual variables occurring freely in A, no free occurrence of α_m $(1 \leqq m \leqq n)$ in A is in a wf part of A of the form $(\beta_m)\Gamma$, and B results from A by replacing all free occurrences of α_1 by β_1, all free occurrences of α_2 by β_2, $\cdots$, all free occurrences of α_n by β_n in A.

$\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_n$A will be called an abstract of B where A and B are as in 2.3.

Rule IV is extended over functional variables of individuals and propositional variables.[4]

It is apparent from the axioms and rules of inference that all the proofs of $S2^1$ and $S4^1$ can be paralleled in the corresponding extended systems for variables of higher type. The same numerals will be used to indicate these parallel proofs.

The systems resulting from the extension of $S2^1$ and $S4^1$ to include the above axioms and rules will be referred to as $S2^2$ and $S4^2$ respectively.

Definitions.

$(\beta \in \mathrm{r}) =_{\mathrm{df}} \mathrm{r}(\beta)$ where α and β are individual variables and r is an abstract of the form $\hat{\alpha}$A.

$(\beta_1 \mathrm{R} \beta_2) =_{\mathrm{df}} \mathrm{R}(\beta_1, \beta_2)$ where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are individual variables and R is an abstract of the form $\hat{\alpha}_1\hat{\alpha}_2$A.

These two definitions will be used without explicit mention in subsequent proofs.

Identity of individuals.

$$\mathrm{I} =_{\mathrm{df}} \hat{\alpha}_1\hat{\alpha}_2(\theta)(\theta(\alpha_1) \prec \theta(\alpha_2)).$$
$$\bar{\mathrm{I}} =_{\mathrm{df}} \hat{\alpha}_1\hat{\alpha}_2\sim(\theta)(\theta(\alpha_1) \prec \theta(\alpha_2)).$$
$$\mathrm{I_m} =_{\mathrm{df}} \hat{\alpha}_1\hat{\alpha}_2(\theta)(\theta(\alpha_1) \supset \theta(\alpha_2)).$$
$$\bar{\mathrm{I}}_{\mathrm{m}} =_{\mathrm{df}} \hat{\alpha}_1\hat{\alpha}_2\sim(\theta)(\theta(\alpha_1) \supset \theta(\alpha_2)).$$

The proofs of theorems 2.4–2.16 are straightforward. Most of them are similar to analogous theorems of *Principia mathematica.*

2.4. $\vdash (\beta_1 \mathrm{I} \beta_2) \prec (\theta)(\theta(\beta_1) \prec \theta(\beta_2))$.
2.5. $\vdash (\beta_1 \mathrm{I} \beta_2) \equiv (\theta)(\theta(\beta_1) \equiv \theta(\beta_2))$.
2.6. $\vdash \beta \mathrm{I} \beta$.
2.7. $\vdash (\exists\beta)(\alpha \mathrm{I} \beta)$.
2.8. $\vdash \sim(\beta_1 \mathrm{I} \beta_2) \equiv (\beta_1 \bar{\mathrm{I}} \beta_2)$.
2.9. $\vdash (\beta_1 \mathrm{I} \beta_2) \equiv (\beta_2 \mathrm{I} \beta_1)$.
2.10. $\vdash (\beta_1 \mathrm{I} \beta_2) \prec (\psi)(\psi(\beta_1, \alpha_1, \alpha_2, \cdots, \alpha_n) \equiv \psi(\beta_2, \alpha_1, \alpha_2, \cdots, \alpha_n))$.
2.11. $\vdash ((\beta_1 \mathrm{I} \beta_2)(\beta_2 \mathrm{I} \beta_3)) \prec (\beta_1 \mathrm{I} \beta_3)$.
2.12. $\vdash ((\beta_1 \mathrm{I} \beta_2)(\beta_2 \bar{\mathrm{I}} \beta_3)) \prec (\beta_1 \bar{\mathrm{I}} \beta_3)$.

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[3] 1.8, extended over propositional variables is a special case of 2.1. 1.8, extended over functional variables is a special case of 2.2.

[4] The following condition should be added to Rule IV on p. 2 of *A functional calculus of first order etc.*: α should not occur freely in A at a place where β would be bound.

2.13. $\vdash (\beta_1 I \beta_2) \prec ((\beta_1 \epsilon \hat{\alpha}A) \equiv (\beta_2 \epsilon \hat{\alpha}A))$.
2.14. $\vdash ((\beta_1 I \beta_2)(\beta_1 \epsilon \hat{\alpha}A)) \prec (\beta_2 \epsilon \hat{\alpha}A)$.
2.15. $\vdash (\beta_1 I \beta_2) \prec (B_1 \equiv B_2)$ where β_1 occurs freely in n places of B_1, and B_2 results from B_1 by replacing m free occurrences of β_1 in B_1 by β_2.
2.16. $\vdash ((\beta_1 I \beta_2)(\Gamma \prec B_1)) \prec (\Gamma \prec B_2)$ where β_1, β_2, B_1, and B_2 are as in 2.15.

The analogous theorems for the relation I_m can readily be established. We will list a few theorems involving this relation for use in subsequent proofs.

2.17. $\vdash (\beta_1 I_m \beta_2) \prec (\theta)(\theta(\beta_1) \supset \theta(\beta_2))$.
2.18. $\vdash \beta I_m \beta$.
2.19. $\vdash (\beta_1 I_m \beta_2) \equiv (\beta_2 I_m \beta_1)$.
2.20. $\vdash \sim(\beta_1 I_m \beta_2) \equiv (\beta_1 \bar{I}_m \beta_2)$.
2.21. $\vdash (\beta_1 I_m \beta_2) \prec ((\beta_1 \epsilon \hat{\alpha}A) \supset (\beta_2 \epsilon \hat{\alpha}A))$.
2.22. $\vdash ((\beta_1 I_m \beta_2)(\beta_1 \epsilon \hat{\alpha}A)) \prec (\beta_2 \epsilon \hat{\alpha}A)$.
2.23. $\vdash \Box(\beta_1 I_m \beta_2) \equiv (\beta_1 I \beta_2)$.
2.24. $\vdash \Box(\beta I_m \beta)$.
2.25. $\vdash (\exists\beta_1)((\beta_1 I \beta_2)(\beta_1 \epsilon \hat{\alpha}A)) \prec (\beta_2 \epsilon \hat{\alpha}A)$ where β_1 is not free in A.
2.14, gen, lemma of 1.87, subst

The converse[5] of 2.25 is not provable in $S2^2$. We can however prove the following rule:

XXX. If $\vdash \beta_2 \epsilon \hat{\alpha}A$ then $\vdash (\exists\beta_1)((\beta_1 I \beta_2)(\beta_1 \epsilon \hat{\alpha}A))$.
hyp, 2.6, adj, 1.16, mod pon

2.26. $\vdash (\exists\beta_1)((\beta_1 I \beta_2)\Box(\beta_1 \epsilon \hat{\alpha}A)) \equiv \Box(\beta_2 \epsilon \hat{\alpha}A)$ where β_1 is not free in A.
$(\exists\beta_1)((\beta_1 I \beta_2)\Box(\beta_1 \epsilon \hat{\alpha}A)) \prec \Box(\beta_2 \epsilon \hat{\alpha}A)$
2.22, VII, *19.81*, subst, 2.23, gen, lemma of 1.87
$(\beta_2 \epsilon \hat{\alpha}A) \prec ((\beta_2 I_m \beta_2)(\beta_2 \epsilon \hat{\alpha}A))$ *12.1*, 2.24, adj, *18.61*, mod pon
$(\exists\beta_1)(\beta_1 I \beta_2)\Box(\beta_1 \epsilon \hat{\alpha}A)) \equiv \Box(\beta_2 \epsilon \hat{\alpha}A)$
VII, *19.81*, subst, 2.23, 1.16, VIII, adj, def

2.27. $\vdash (\beta_1)((\beta_1 I_m \beta_2) \supset (\beta_1 \epsilon \hat{\alpha}A)) \equiv (\beta_2 \epsilon \hat{\alpha}A)$ where β_1 is not free in A.
$(\beta_1)((\beta_1 I_m \beta_2) \supset (\beta_1 \epsilon \hat{\alpha}A)) \prec (\beta_2 \epsilon \hat{\alpha}A)$
1.8, *14.26*, subst, *12.15*, 2.24, adj, *18.61*, mod pon
$(\beta_1)((\beta_1 I_m \beta_2) \supset (\beta_1 \epsilon \hat{\alpha}A)) \equiv (\beta_2 \epsilon \hat{\alpha}A)$
2.19, 2.21, subst, *14.26*, XVII adj, def

2.28. $\vdash (\beta_1)((\beta_1 I_m \beta_2) \prec (\beta_1 \epsilon \hat{\alpha}A)) \equiv \Box(\beta_2 \epsilon \hat{\alpha}A)$ where β_1 is not free in A.
2.27, VII, *18.7*, 1.39, subst

2.29. $\vdash \sim\Diamond(\beta_2 \epsilon \hat{\alpha}A) \prec (\beta_1)((\beta_1 \epsilon \hat{\alpha}A) \prec (\beta_1 \bar{I} \beta_2))$ where β_1 is not free in A.
$\sim(\beta_2 \epsilon \hat{\alpha}A) \prec (\beta_1)(\sim(\beta_1 \epsilon \hat{\alpha}A) \vee (\beta_1 \bar{I} \beta_2))$
2.25, X, *14.21*, subst, *13.11*, def, 2.8
$\sim\Diamond(\beta_2 \epsilon \hat{\alpha}A) \prec (\beta_1)((\beta_1 \epsilon \hat{\alpha}A) \prec (\beta_1 \bar{I} \beta_2))$
VII, *12.3*, 1.39, subst, *14.2*, *18.7*

[5] To prove the converse of such a theorem as 2.25 requires some principle such as $\Box A \prec (B \prec \Box A)$. That this principle is not provable in $S2^2$ can be shown by the Group I matrix on p. 493 of Lewis and Langford's *Symbolic Logic*, where this matrix has been appropriately interpreted for quantification.

The converse of 2.29 is not provable in $S2^2$. We are able however to prove the following rule:

XXXI. If $\vdash (\beta_1)((\beta_1 \in \hat{\alpha}A) \prec (\beta_1 \bar{I} \beta_2))$ then $\vdash \sim\Diamond(\beta_2 \in \hat{\alpha}A)$.
$\sim\Diamond((\beta_1 \in \hat{\alpha}A)(\beta_1 I \beta_2))$ hyp, 1.8, mod pon, def, 2.8, *12.3*, subst
$((\beta_1 \in \hat{\alpha}A)(\beta_1 I \beta_2)) \prec \sim(\beta_2 \in \hat{\alpha}A)$ *19.74*, mod pon
$((\beta_1 \in \hat{\alpha}A)(\beta_1 I \beta_2)) \prec (\beta_2 \in \hat{\alpha}A)$ 2.14, *12.15*, subst
$\sim\Diamond(\beta_2 \in \hat{\alpha}A)$ adj, *19.72*, subst

Where the identity relation appearing in 2.29 is weakened to I_m we can prove the following:

2.30. $\vdash (\beta_1)((\beta_1 \in \hat{\alpha}A) \prec (\beta_1 \bar{I}_m \beta_2)) \equiv \sim\Diamond(\beta_2 \in \hat{\alpha}A)$ where β_1 is not free in A.
2.28, 2.3, subst, *12.3*

The material equivalence of I and I_m can be proved in $S2^2$.

2.31. $\vdash (\beta_1 I_m \beta_2) \equiv (\beta_1 I \beta_2)$.
$(\beta_1 I_m \beta_2) \supset ((\beta_1 I \beta_1) \supset (\beta_1 I \beta_2))$ 2.21, *14.1*, mod pon, 2.3, subst
$(\beta_1 I_m \beta_2) \supset (\beta_1 I \beta_2)$ *15.8*, subst, 2.6, adj, *14.29*, mod pon
$(\beta_1 I \beta_2) \supset (\beta_1 I_m \beta_2)$ *12.1*, 2.23, subst, *18.42*, VIII, *14.1*, mod pon
$(\beta_1 I_m \beta_2) \equiv (\beta_1 I \beta_2)$ adj, def

In $S4^2$ we can prove the strict equivalence of I and I_m.

2.32*. $\vdash \Box(\beta_1 I \beta_2)(\beta_1 I \beta_2)$. 2.23, 1.104*, subst

2.33*. $\vdash (\beta_1 I \beta_2) \equiv (\beta_1 I_m \beta_2)$.
$((\beta_1 I_m \beta_2)(\beta_1 I \beta_1)) \prec (\beta_1 I \beta_2)$ 2.21, 2.3, subst, *14.26*
$(\beta_1 I_m \beta_2) \prec (\beta_1 I \beta_2)$ 2.6, 2.32*, subst, adj, *18.61*, mod pon
$(\beta_1 I \beta_2) \equiv (\beta_1 I_m \beta_2)$ *18.42*, 2.23, subst, adj, def

A direct consequence of 2.33* is

2.34*. $\vdash (\beta_1 I_m \beta_2) \prec (B_1 \equiv B_2)$ where β_1, β_2, B_1, B_2 are as in 2.15.

We can also derive without modification such properties of identity as:

2.35*. $\vdash (\exists\beta_1)((\beta_1 I \beta_2)(\beta_1 \in \hat{\alpha}A)) \equiv (\beta_2 \in \hat{\alpha}A)$ where β_1 is not free in A.
12.1, 2.6, 2.32*, adj, *18.61*, mod pon, 1.16, VIII, 2.25, def

2.36*. $\vdash \Box(\beta_2 \in \hat{\alpha}A) \equiv (\beta_1)((\beta_1 I \beta_2) \prec (\beta_1 \in \hat{\alpha}A))$ where β_1 is not free in A.
2.28, 2.33*, subst

2.37*. $\vdash (\beta_1)((\beta_1 \in \hat{\alpha}A) \prec (\beta_1 \bar{I} \beta_2)) \equiv \sim\Diamond(\beta_2 \in \hat{\alpha}A)$ where β_1 is not free in A.
2.30, 2.20, 2.8, 2.33* subst

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