On the Algebra of Logic
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On the Algebra of Logic.

BY C. S. PEIRCE.

CHAPTER I. — SYLLOGISTIC.

§ 1. *Derivation of Logic.*

IN order to gain a clear understanding of the origin of the various signs used in logical algebra and the reasons of the fundamental formulæ, we ought to begin by considering how logic itself arises.

Thinking, as cerebration, is no doubt subject to the general laws of nervous action.

When a group of nerves are stimulated, the ganglions with which the group is most intimately connected on the whole are thrown into an active state, which in turn usually occasions movements of the body. The stimulation continuing, the irritation spreads from ganglion to ganglion (usually increasing meantime). Soon, too, the parts first excited begin to show fatigue; and thus for a double reason the bodily activity is of a changing kind. When the stimulus is withdrawn, the excitement quickly subsides.

It results from these facts that when a nerve is affected, the reflex action, if it is not at first of the sort to remove the irritation, will change its character again and again until the irritation is removed; and then the action will cease.

Now, all vital processes tend to become easier on repetition. Along whatever path a nervous discharge has once taken place, in that path a new discharge is the more likely to take place.

Accordingly, when an irritation of the nerves is repeated, all the various actions which have taken place on previous similar occasions are the more likely to take place now, and those are most likely to take place which have most frequently taken place on those previous occasions. Now, the various actions which did not remove the irritation may have previously sometimes been performed and sometimes not; but the action which removes the irritation must have always been performed, because the action must have every time continued until it was performed. Hence, a strong habit of responding to the given irritation in this particular way must quickly be established.

A habit so acquired may be transmitted by inheritance.

One of the most important of our habits is that one by virtue of which certain classes of stimuli throw us at first, at least, into a purely cerebral activity.

Very often it is not an outward sensation but only a fancy which starts the train of thought. In other words, the irritation instead of being peripheral is visceral. In such a case the activity has for the most part the same character; an inward action removes the inward excitation. A fancied conjuncture leads us to fancy an appropriate line of action. It is found that such events, though no external action takes place, strongly contribute to the formation of habits of really acting in the fancied way when the fancied occasion really arises.

A cerebral habit of the highest kind, which will determine what we do in fancy as well as what we do in action, is called a *belief*. The representation to ourselves that we have a specified habit of this kind is called a *judgment*. A belief-habit in its development begins by being vague, special, and meagre; it becomes more precise, general, and full, without limit. The process of this development, so far as it takes place in the imagination, is called *thought*. A judgment is formed; and under the influence of a belief-habit this gives rise to a new judgment, indicating an addition to belief. Such a process is called an *inference;* the antecedent judgment is called the *premise;* the consequent judgment, the *conclusion;* the habit of thought, which determined the passage from the one to the other (when formulated as a proposition), the *leading principle.*

At the same time that this process of inference, or the spontaneous development of belief, is continually going on within us, fresh peripheral excitations are also continually creating new belief-habits. Thus, belief is partly determined by old beliefs and partly by new experience. Is there any law about the mode of the peripheral excitations? The logician maintains that there is, namely, that they are all adapted to an end, that of carrying belief, in the long run, toward certain predestinate conclusions which are the same for all men. This is the faith of the logician. This is the matter of fact, upon which all maxims of reasoning repose. In virtue of this fact, what is to be believed at last is independent of what has been believed hitherto, and therefore has the character of *reality*. Hence, if a given habit, considered as determining an inference, is of such a sort as to tend toward the final result, it is correct; otherwise not. Thus, inferences become divisible into the valid and the invalid; and thus logic takes its reason of existence.

§ 2. *Syllogism and Dialogism.*

The general type of inference is

$$\begin{array}{c} \text{P} \\ \therefore \text{C}, \end{array}$$

where $\therefore$ is the sign of illation.

The passage from the premise (or set of premises) P to the conclusion C takes place according to a habit or rule active within us. All the inferences which that habit would determine when once the proper premises were admitted, form a class. The habit is logically good provided it would never (or in the case of a probable inference, seldom) lead from a true premise to a false conclusion; otherwise it is logically bad. That is, every possible case of the operation of a good habit would either be one in which the premise was false or one in which the conclusion would be true; whereas, if a habit of inference is bad, there is a possible case in which the premise would be true, while the conclusion was false. When we speak of a *possible* case, we conceive that from the general description of cases we have struck out all those kinds which we know how to describe in general terms but which we know never will occur; those that then remain, embracing all whose non-occurrence we are not certain of, together with all those whose non-occurrence we cannot explain on any general principle, are called possible.

A habit of inference may be formulated in a proposition which shall state that every proposition c, related in a given general way to any true proposition p, is true. Such a proposition is called the *leading principle* of the class of inferences whose validity it implies. When the inference is first drawn, the leading principle is not present to the mind, but the habit it formulates is active in such a way that, upon contemplating the believed premise, by a sort of perception the conclusion is judged to be true.* Afterwards, when the inference is subjected to logical criticism, we make a new inference, of which one premise is that leading principle of the former inference, according to which propositions related to one another in a certain way are fit to be premise and conclusion of a valid inference, while another premise is a fact of observation, namely, that the given relation does subsist between the premise and conclusion of the inference under criticism; whence it is concluded that the inference was valid.

Logic supposes inferences not only to be drawn, but also to be subjected to criticism; and therefore we not only require the form P $\therefore$ C to express an argu-

* Though the leading principle itself is not present to the mind, we are generally conscious of inferring on some general principle.

ment, but also a form, $P_i -\!\!< C_i$, to express the truth of its leading principle. Here P_i denotes any one of the class of premises, and C_i the corresponding conclusion. The symbol $-\!\!<$ is the copula, and signifies primarily that every state of things in which a proposition of the class P_i is true is a state of things in which the corresponding propositions of the class C_i are true. But logic also supposes some inferences to be invalid, and must have a form for denying the leading premise. This we shall write $P_i \overline{-\!\!<} C_i$, *a dash over any symbol signifying in our notation the negative of that symbol.**

Thus, the form $P_i -\!\!< C_i$ implies

either, 1, that it is impossible that a premise of the class P_i should be true,

or, 2, that every state of things in which P_i is true is a state of things in which the corresponding C_i is true.

The form $P_i \overline{-\!\!<} C_i$ implies

both, 1, that a premise of the class P_i is possible,

and, 2, that among the possible cases of the truth of a P_i there is one in which the corresponding C_i is not true.

This acceptation of the copula differs from that of other systems of syllogistic in a manner which will be explained below in treating of the negative.

In the form of inference $P \therefore C$ the leading principle is not expressed; and the inference might be justified on several separate principles. One of these, however, $P_i -\!\!< C_i$, is the formulation of the habit which, in point of fact, has governed the inferences. This principle contains all that is necessary besides the premise P to justify the conclusion. (It will generally assert more than is necessary.) We may, therefore, construct a new argument which shall have for its premises the two propositions P and $P_i -\!\!< C_i$ taken together, and for its conclusion, C. This argument, no doubt, has, like every other, its leading principle, because the inference is governed by some habit; but yet the substance of the leading principle must already be contained implicitly in the premises, because the proposition $P_i -\!\!< C_i$ contains by hypothesis all that is requisite to justify the inference of C from P. Such a leading principle, which contains no fact not implied or observable in the premises, is termed a *logical* principle, and the argument it governs is termed a *complete*, in contradistinction to an *incomplete*, argument, or *enthymeme*.

The above will be made clear by an example. Let us begin with the enthymeme,

Enoch was a man,
$\therefore$ Enoch died.

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* This dash was used by Boole, but not over other than class-signs.

The leading principle of this is, "All men die." Stating it, we get the complete argument,

All men die,
Enoch was a man;
∴ Enoch was to die.

The leading principle of this is *nota notae est nota rei ipsius*. Stating this as a premise, we have the argument,

Nota notae est nota rei ipsius,
Mortality is a mark of humanity, which is a mark of Enoch;
∴ Mortality is a mark of Enoch.

But this very same principle of the *nota notae* is again active in the drawing of this last inference, so that the last state of the argument is no more complete than the last but one.

There is another way of rendering an argument complete, namely, instead of adding the leading principle $P_i \mathrel{-\!\!<} C_i$ conjunctively to the premise P, to form a new argument, we might add its denial disjunctively to the conclusion; thus,

$$\begin{gathered} P \\ \therefore \text{ Either } C \text{ or } P_i \mathrel{\overline{-\!\!<}} C_i\,. \end{gathered}$$

A logical principle is said to be an *empty* or merely formal proposition, because it can add nothing to the premises of the argument it governs, although it is relevant; so that it implies no fact except such as is presupposed in all discourse, as we have seen in § 1 that certain facts are implied. We may here distinguish between *logical* and *extralogical* validity; the former being that of a *complete*, the latter that of an *incomplete* argument. The term *logical leading principle* we may take to mean the principle which must be supposed true in order to sustain the logical validity of any argument. Such a principle states that among all the states of things which can be supposed without conflict with logical principles, those in which the premise of the argument would be true would also be cases of the truth of the conclusion. Nothing more than this would be relevant to the *logical leading principle*, which is, therefore, perfectly determinate and not vague, as we have seen an extralogical leading principle to be.

A complete argument, with only one premise, is called an *immediate* inference. *Example:* All crows are black birds; therefore, all crows are birds. If from the premise of such an argument everything redundant is omitted, the state of things expressed in the premise is the same as the state of things expressed in the conclusion, and only the form of expression is changed. Now, the logician does not undertake to enumerate all the ways of expressing facts:

he supposes the facts to be already expressed in certain standard or canonical forms. But the equivalence between different ones of his own standard forms is of the highest importance to him, and thus certain immediate inferences play the great part in formal logic. Some of these will not be reciprocal inferences or logical equations, but the most important of them will have that character.

If one fact has such a relation to a different one that, if the former be true, the latter is necessarily or probably true, this relation constitutes a determinate fact; and therefore, since the leading principle of a complete argument involves no matter of fact (beyond those employed in all discourse), it follows that every complete and *material* (in opposition to a merely *formal*) argument must have at least two premises.

From the doctrine of the leading principle it appears that if we have a valid and complete argument from more than one premise, we may suppress all premises but one and still have a valid but incomplete argument. This argument is justified by the suppressed premises; hence, from these premises alone we may infer that the conclusion would follow from the remaining premises. In this way, then, the original argument

$$\begin{array}{c} P\ Q\ R\ S\ T \\ \therefore C \end{array}$$

is broken up into two, namely, 1st,

$$\begin{array}{c} P\ Q\ R\ S \\ \therefore T -\!\!< C \end{array}$$

and, 2d,

$$\begin{array}{c} T -\!\!< C \\ T \\ \therefore C. \end{array}$$

By repeating this process, any argument may be broken up into arguments of two premises each. A complete argument having two premises is called a *syllogism*.*

An argument may also be broken up in a different way by substituting for the second constituent above, the form

$$\begin{array}{c} T -\!\!< C \\ \therefore \text{Either } C \text{ or not } T. \end{array}$$

In this way, any argument may be resolved into arguments, each of which has one premise and two alternative conclusions. Such an argument, when complete, may be called a *dialogism*.

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* The general doctrine of this section is contained in my paper, *On the Classification of Arguments*, 1867.

§ 3. *Forms of Propositions.*

In place of the two expressions A $-\!\!<$ B and B $-\!\!<$ A taken together we may write A $=$ B;* in place of the two expressions A $-\!\!<$ B and B $\overline{-\!\!<}$ A taken together we may write A $<$ B or B $>$ A; and in place of the two expressions A $\overline{-\!\!<}$ B and B $\overline{-\!\!<}$ A taken together we may write A $\asymp$ B.

De Morgan, in the remarkable memoir with which he opened his discussion of the syllogism (1846, p. 380), has pointed out that we often carry on reasoning under an implied restriction as to what we shall consider as possible, which restriction, applying to the whole of what is said, need not be expressed. The total of all that we consider possible is called the *universe* of discourse, and may be very limited. One mode of limiting our universe is by considering only what actually occurs, so that everything which does not occur is regarded as impossible.

The forms A $-\!\!<$ B, or A implies B, and A $\overline{-\!\!<}$ B, or A does not imply B, embrace both hypothetical and categorical propositions. Thus, to say that all men are mortal is the same as to say that if any man possesses any character whatever then a mortal possesses that character. To say, 'if A, then B' is obviously the same as to say that from A, B follows, logically or extralogically. By thus identifying the relation expressed by the copula with that of illation,

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* There is a difference of opinion among logicians as to whether $-\!\!<$ or $=$ is the simpler relation. But in my paper on the *Logic of Relatives*, I have strictly demonstrated that the preference must be given to $-\!\!<$ in this respect. The term *simpler* has an exact meaning in logic; it means that whose logical depth is smaller; that is, if one conception implies another, but not the reverse, then the latter is said to be the simpler. Now to say that A $=$ B implies that A $-\!\!<$ B, but not conversely. *Ergo*, etc. It is to no purpose to reply that A $-\!\!<$ B implies A $=$ (A that is B); it would be equally relevant to say that A $-\!\!<$ B implies A $=$ A. Consider an analogous case. Logical sequence is a simpler conception than causal sequence, because every causal sequence is a logical sequence but not every logical sequence is a causal sequence; and it is no reply to this to say that a logical sequence between two facts implies a causal sequence between some two facts whether the same or different. The idea that $=$ is a very simple relation is probably due to the fact that the discovery of such a relation teaches us that instead of two objects we have only one, so that it simplifies our conception of the universe. On this account the existence of such a relation is an important fact to learn; in fact, it has the sum of the importances of the two facts of which it is compounded. It frequently happens that it is more convenient to treat the propositions A $-\!\!<$ B and B $-\!\!<$ A together in their form A $=$ B; but it also frequently happens that it is more convenient to treat them separately. Even in geometry we can see that to say that two figures A and B are equal is to say that when they are properly put together A will cover B and B will cover A; and it is generally necessary to examine these facts separately. So, in comparing the numbers of two lots of objects, we set them over against one another, each to each, and observe that for every one of the lot A there is one of the lot B, and for every one of the lot B there is one of the lot A.

In logic, our great object is to analyze all the operations of reason and reduce them to their ultimate elements; and to make a calculus of reasoning is a subsidiary object. Accordingly, it is more philosophical to use the copula $-\!\!<$, apart from all considerations of convenience. Besides, this copula is intimately related to our natural logical and metaphysical ideas; and it is one of the chief purposes of logic to show what validity those ideas have. Moreover, it will be seen further on that the more analytical copula does in point of fact give rise to the easiest method of solving problems of logic.

we identify the proposition with the inference, and the term with the proposition. This identification, by means of which all that is found true of term, proposition, or inference is at once known to be true of all three, is a most important engine of reasoning, which we have gained by beginning with a consideration of the genesis of logic.*

Of the two forms $A \prec B$ and $A \overline{\prec} B$, no doubt the former is the more primitive, in the sense that it is involved in the idea of reasoning, while the latter is only required in the criticism of reasoning. The two kinds of proposition are essentially different, and every attempt to reduce the latter to a special case of the former must fail. Boole attempts to express 'some men are not mortal,' in the form 'whatever men have a certain unknown character v are not mortal.' But the propositions are not identical, for the latter does not imply that some men have that character v; and, accordingly, from Boole's proposition we may legitimately infer that 'whatever mortals have the unknown character v are not men'; yet we cannot reason from 'some men are not mortal' to 'some mortals are not men.'† On the other hand, we can rise to a more general form under which $A \prec B$ and $A \overline{\prec} B$ are both included. For this purpose we write $A \overline{\prec} B$ in the form $\breve{A} \prec \bar{B}$, where $\breve{A}$ is *some*-A and $\bar{B}$ is *not*-B. This more general form is equivocal in so far as it is left undetermined whether the proposition would be true if the subject were impossible. When the subject is general this is the case, but when the subject is particular (i. e., is subject to the modification *some*) it is not. The general form supposes merely inclusion of the subject under the predicate. The short curved mark over the letter in the subject shows that some part of the term denoted by that letter is the subject, and that that is asserted to be in possible existence.

The modification of the subject by the curved mark and of the predicate by the straight mark gives the old set of propositional forms, viz.:

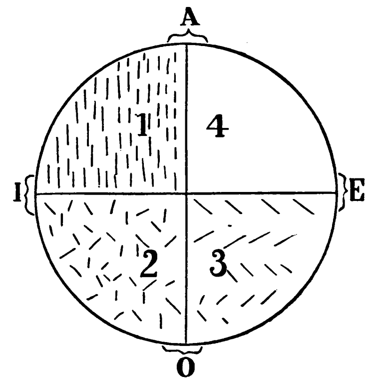
A.	$a \prec b$	Every a is b.	Universal affirmative.
E.	$a \prec \bar{b}$	No a is b.	Universal negative.
I.	$\breve{a} \prec b$	Some a is b.	Particular affirmative.
O.	$\breve{a} \prec \bar{b}$	Some a is not b.	Particular negative.

There is, however, a difference between the senses in which these propo-

* In consequence of the identification in question, in $S \prec P$, I speak of S indifferently as *subject*, *antecedent*, or *premise*, and of P as *predicàte*, *consequent*, or *conclusion*.

† Equally unsuccessful is Mr. Jevons's attempt to overcome the difficulty by omitting particular propositions, 'because we can always substitute for it [*some*] more definite expressions if we like.' The same reason might be alleged for neglecting the consideration of *not*. But in fact the form $A \overline{\prec} B$ is required to enable us to simply deny $A \prec B$.

sitions are here taken and those which are traditional; namely, it is usually understood that affirmative propositions imply the existence of their subjects, while negative ones do not. Accordingly, it is said that there is an immediate inference from A to I and from E to O. But in the sense assumed in this paper, universal propositions do not, while particular propositions do, imply the existence of their subjects. The following figure illustrates the precise sense here assigned to the four forms A, E, I, O.



In the quadrant marked 1 there are lines which are all vertical; in the quadrant marked 2 some lines are vertical and some not; in quadrant 3 there are lines none of which are vertical; and in quadrant 4 there are no lines. Now, taking *line* as subject and *vertical* as predicate,

A is true of quadrants 1 and 4 and false of 2 and 3.
E is true of quadrants 3 and 4 and false of 1 and 2.
I is true of quadrants 1 and 2 and false of 3 and 4.
O is true of quadrants 2 and 3 and false of 1 and 4.

Hence, A and O precisely deny each other, and so do E and I. But any other pair of propositions may be either both true or both false or one true while the other is false.

De Morgan (On the Syllogism, No. I., 1846, p. 381) has enlarged the system of propositional forms by applying the sign of negation which first appears in $A \,\overline{\prec}\, B$ to the subject and predicate. He thus gets

$A \prec B$.	Every A is B.	A is species of B.
$A \,\overline{\prec}\, B$.	Some A is not B.	A is exient of B.
$A \prec \bar{B}$.	No A is B.	A is external of B.
$A \,\overline{\prec}\, \bar{B}$.	Some A is B.	A is partient of B.
$\bar{A} \prec B$.	Everything is either A or B.	A is complement of B.
$\bar{A} \,\overline{\prec}\, B$.	There is something besides A and B.	A is coinadequate of B.
$\bar{A} \prec \bar{B}$.	A includes all B.	A is genus of B.
$\bar{A} \,\overline{\prec}\, \bar{B}$.	A does not include all B.	A is deficient of B.

De Morgan's table of the relations of these propositions must be modified to conform to the meanings here attached to $\prec$ and to $\overline{\prec}$.

We might confine ourselves to the two propositional forms $S \prec P$ and $S \,\overline{\prec}\, P$. If we once go beyond this and adopt the form $S \prec \bar{P}$, we must, for the sake of completeness, adopt the whole of De Morgan's system. But this system, as we shall see in the next section, is itself incomplete, and requires to complete it the admission of particularity in the predicate. This has already been attempted by Hamilton, with an incompetence which ought to be extraordinary. I shall allude to this matter further on, but I shall not attempt to say how many forms of propositions there would be in the completed system.*

§ 4. *The Algebra of the Copula.*

From the identity of the relation expressed by the copula with that of illation, springs an algebra. In the first place, this gives us

$$x \prec x \tag{1}$$

the principle of identity, which is thus seen to express that what we have hitherto believed we continue to believe, in the absence of any reason to the contrary. In the next place, this identification shows that the two inferences

$$\begin{array}{ccc} x & & \\ y & \text{and} & x \\ \therefore z & & \therefore y \prec z \end{array} \tag{2}$$

are of the same validity. Hence we have

$$\{x \prec (y \prec z)\} = \{y \prec (x \prec z)\}.\dagger \tag{3}$$

From (1) we have

$$(x \prec y) \prec (x \prec y),$$

whence by (2)

$$\begin{array}{cc} x \prec y & x \\ \multicolumn{2}{c}{\therefore y} \end{array} \tag{4}$$

is a valid inference.

By (4), if x and $x \prec y$ are true y is true; and if y and $y \prec z$ are true z is true. Hence, the inference is valid

$$\begin{array}{ccc} x & x \prec y & y \prec z \\ & \therefore z. & \end{array}$$

By the principle of (2) this is the same as to say that

$$\begin{array}{cc} x \prec y & y \prec z \\ \multicolumn{2}{c}{\therefore x \prec z} \end{array} \tag{5}$$

is a valid inference. This is the canonical form of the syllogism, *Barbara*. The

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* In this connection see De Morgan, *On the Syllogism*, No. V., 1862.

† Mr. Hugh McColl (*Calculus of Equivalent Statements*, Second Paper, 1878, p. 183) makes use of the sign of inclusion several times in the same proposition. He does not, however, give any of the formulæ of this section.

statement of its validity has been called the *dictum de omni*, the *nota notae*, etc.; but it is best regarded, after De Morgan,* as a statement that the relation signified by the copula is a transitive one.† It may also be considered as implying that in place of the subject of a proposition of the form $A \prec B$, any subject of that subject may be substituted, and that in place of its predicate any predicate of that predicate may be substituted.‡ The same principle may be algebraically conceived as a rule for the elimination of y from the two propositions $x \prec y$ and $y \prec z$.§

It is needless to remark that any letters may be substituted for x, y, z; and that, therefore, $\bar{x}$, $\bar{y}$, $\bar{z}$, some or all, may be substituted. Nevertheless, after these purely extrinsic changes have been made, the argument is no longer called *Barbara*, but is said to be some other universal mood of the *first figure*. There are evidently eight such moods.

From (5) we have, by (2), these two forms of valid immediate inference:

$$\begin{array}{c} S \prec P \\ \therefore\ (x \prec S) \prec (x \prec P) \end{array} \tag{6}$$

and

$$\begin{array}{c} S \prec P \\ \therefore\ (P \prec x) \prec (S \prec x). \end{array} \tag{7}$$

The latter may be termed the inference of *contraposition.*

From the transitiveness of the copula, the following inference is valid:

$$\begin{array}{l} (S \prec M) \prec (S \prec P) \\ (S \prec P) \prec x \\ \therefore\ (S \prec M) \prec x. \end{array}$$

But, by (6), from $(M \prec P)$ we can infer the first premise immediately; hence the inference is valid

$$\begin{array}{c} M \prec P \\ (S \prec P) \prec x \\ \therefore\ (S \prec M) \prec x. \end{array} \tag{8}$$

* *On the Syllogism*, No. II., 1850, p. 104.

† That the validity of syllogism is not deducible from the principles of identity, contradiction, and excluded middle, is capable of strict demonstration. The transitiveness of the copula is, however, implied in the identification of the copula-relation with illation, because illation is obviously transitive.

‡ The conception of substitution (already involved in the mediæval doctrine of descent), as well as the word, was familiar to logicians before the publication of Mr. Jevons's *Substitution of Similars*. This book argues, however, not only that inference is substitution, but that it and induction in particular consist in the substitution of similars. This doctrine is allied to Mill's theory of induction.

§ This must have been in Boole's mind from the first. De Morgan (*On the Syllogism*, No. II., 1850, p. 83) goes too far in saying that "what is called elimination in algebra is called inference in logic," if he means, as he seems to do, that all inference is elimination.

This may be called the *minor indirect syllogism.* The following is an example:

All men are mortal,
If Enoch and Elijah were mortal, the Bible errs;
∴ If Enoch and Elijah were men, the Bible errs.

Again we may start with this syllogism in *Barbara*

$$\begin{aligned} &(\mathrm{M} \prec \mathrm{P}) \prec (\mathrm{S} \prec \mathrm{P}), \\ &(\mathrm{S} \prec \mathrm{P}) \prec x; \\ \therefore\ &(\mathrm{M} \prec \mathrm{P}) \prec x. \end{aligned}$$

But by the principle of contraposition (7), the first premise immediately follows from ($\mathrm{S} \prec \mathrm{M}$), so that we have the inference valid

$$\begin{aligned} &\mathrm{S} \prec \mathrm{M}, \\ &(\mathrm{S} \prec \mathrm{P}) \prec x; \\ \therefore\ &(\mathrm{M} \prec \mathrm{P}) \prec x. \end{aligned} \tag{9}$$

This may be called the *major indirect syllogism.*

Example: All patriarchs are men,
If all patriarchs are mortal, the Bible errs;
∴ If all men are mortal, the Bible errs.

In the same way it might be shown that (6) justifies the syllogism

$$\begin{aligned} &\mathrm{M} \prec \mathrm{P}, \\ &x \prec (\mathrm{S} \prec \mathrm{M}); \\ \therefore\ &x \prec (\mathrm{S} \prec \mathrm{P}). \end{aligned} \tag{10}$$

And (7) justifies the inference

$$\begin{aligned} &\mathrm{S} \prec \mathrm{M}, \\ &x \prec (\mathrm{M} \prec \mathrm{P}); \\ \therefore\ &x \prec (\mathrm{S} \prec \mathrm{P}). \end{aligned} \tag{11}$$

But these are only slight modifications of *Barbara.*

In the form (10), x may denote a limited universe comprehending some cases of S. Then we have the syllogism

$$\begin{aligned} &\mathrm{M} \prec \mathrm{P}, \\ &\mathrm{S}\ \overline{\prec}\ \bar{\mathrm{M}}; \\ \therefore\ &\mathrm{S}\ \overline{\prec}\ \bar{\mathrm{P}}. \end{aligned} \tag{12}$$

This is called *Darii.* A line might, of course, be drawn over the S. So, in the form (11), x may denote a limited universe comprehending some $\bar{\mathrm{M}}$. Then we have the syllogism

$$
\begin{aligned}
&\mathrm{S} \prec \mathrm{M}, \\
&\overline{\mathrm{M}} \mathrel{\overline{\prec}} \mathrm{P}; \\
\therefore\ &\overline{\mathrm{S}} \mathrel{\overline{\prec}} \mathrm{P}.
\end{aligned}
\tag{13}
$$

Here a line might be drawn over the P. But the forms (12) and (13) are deduced from (10) and (11) only by principles of interpretation which require demonstration.

On the other hand, if in the *minor indirect syllogism* (8), we put "what does not occur" for x, we have by definition

$$\{(\mathrm{S} \prec \mathrm{P}) \prec x\} = (\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P})$$

and we then have

$$
\begin{aligned}
&\mathrm{M} \prec \mathrm{P}, \\
&\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}; \\
\therefore\ &\mathrm{S} \mathrel{\overline{\prec}} \mathrm{M},
\end{aligned}
\tag{14}
$$

which is the syllogism *Baroko.* If a line is drawn over P, the syllogism is called *Festino;* and by other negations eight essentially identical forms are obtained, which are called minor-particular moods of the second figure.* In the same way the major indirect syllogism (9) affords the form

$$
\begin{aligned}
&\mathrm{S} \prec \mathrm{M}, \\
&\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}; \\
\therefore\ &\mathrm{M} \mathrel{\overline{\prec}} \mathrm{P}.
\end{aligned}
\tag{15}
$$

This form is called *Bocardo.* If P is negatived, it is called *Disamis.* Other negations give the eight major-particular moods of the third figure.

We have seen that $\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}$ is of the form $(\mathrm{S} \prec \mathrm{P}) \prec x$. Put A for $\mathrm{S} \prec \mathrm{P}$, and we find that $\overline{\mathrm{A}}$ is of the form $\mathrm{A} \prec x$. Then the principle of contraposition (7) gives the immediate inference

$$
\begin{aligned}
&\mathrm{S} \prec \mathrm{P} \\
\therefore\ &\overline{\mathrm{P}} \prec \overline{\mathrm{S}}.
\end{aligned}
\tag{16}
$$

Applying this to the universal moods of the first figure justifies six moods. These are two in the second figure,

$$
\begin{array}{lll}
x \prec \bar{y} & z \prec y & \therefore\ x \prec \bar{z} \quad (\textit{Camestres}) \\
\bar{x} \prec \bar{y} & z \prec y & \therefore\ \bar{x} \prec \bar{z}\,;
\end{array}
$$

two in the third figure,

$$
\begin{array}{lll}
y \prec x & \bar{y} \prec z & \therefore\ \bar{x} \prec z \\
y \prec x & \bar{y} \prec \bar{z} & \therefore\ \bar{x} \prec \bar{z}\,;
\end{array}
$$

* De Morgan, *Syllabus*, 1860, p. 18.

and two others which are said to be in the fourth figure,

$$x \prec y \qquad y \prec z \qquad \therefore\ \bar{z} \prec \bar{x}$$
$$x \prec \bar{y} \qquad \bar{y} \prec z \qquad \therefore\ \bar{z} \prec \bar{x}.$$

But the negative has two other properties not yet taken into account. These are

$$x \prec \bar{\bar{x}} \tag{17}$$

or x is not not-X, which is called the *principle of contradiction;* and

$$\bar{\bar{x}} \prec x \tag{18}$$

or what is not not-X is x, which is called the *principle of excluded middle.*

By (17) and (16) we have the immediate inference

$$\begin{array}{c} \mathrm{S} \prec \overline{\mathrm{P}} \\ \therefore\ \mathrm{P} \prec \overline{\mathrm{S}} \end{array} \tag{19}$$

which is called the conversion of E. By (18) and (16) we have

$$\begin{array}{c} \overline{\mathrm{S}} \prec \mathrm{P} \\ \therefore\ \overline{\mathrm{P}} \prec \mathrm{S}. \end{array} \tag{20}$$

By (17), (18), and (16), we have

$$\begin{array}{c} \overline{\mathrm{S}} \prec \overline{\mathrm{P}} \\ \therefore\ \mathrm{P} \prec \mathrm{S}. \end{array} \tag{21}$$

Each of the inferences (19), (20), (21), justifies six universal syllogisms; namely, two in each of the figures, second, third, and fourth. The result is that each of these figures has eight universal moods; two depending only on the principle that $\bar{\mathrm{A}}$ is of the form $\mathrm{A} \prec x$, two depending also on the principle of contradiction, two on the principle of excluded middle, and two on all three principles conjoined.

The same formulæ (16), (19), (20), (21), applied to the minor-particular moods of the second figure, will give eight minor-particular moods of the first figure; and applied to the major-particular moods of the third figure, will give eight major-particular moods of the first figure.*

The principle of contradiction in the form (19) may be further transformed thus: —

$$\text{If } (\mathrm{P} \therefore \overline{\mathrm{C}}) \text{ is valid, then } (\mathrm{C} \therefore \overline{\mathrm{P}}) \text{ is valid.} \tag{22}$$

Applying this to the minor-particular moods of the first figure, will give eight minor-particular moods of the third figure; and applying it to the major-particu-

* Aristotle and De Morgan have particular conclusions from two universal premises. These are all rendered illogical by the significations which I attach to $\prec$ and $\overline{\prec}$.

lar moods of the first figure will give eight major-particular moods of the second figure.

It is very noticeable that the corresponding formula,

$$\text{If } (\bar{P} \therefore C) \text{ is valid, then } (\bar{C} \therefore P) \text{ is valid,} \tag{23}$$

has no application in the existing syllogistic, because there are no syllogisms having a particular premise and universal conclusion. In the same way, in the Aristotelian system an affirmative conclusion cannot be drawn from negative premises, the reason being that negation is only applied to the predicate. So in De Morgan's system the subject only is made particular, not the predicate.

In order to develop a system of propositions in which the predicate shall be modified in the same way in which the subject is modified in particular propositions, we should consider that to say $S \prec P$ is the same as to say $(S \overline{\prec}\, x) \prec (P \overline{\prec}\, x)$, whatever x may be. That

$$(S \prec P) \prec \{(S \overline{\prec}\, x) \prec (P \overline{\prec}\, x)\}$$

follows at once from *Bokardo* (15) by means of (2). Moreover, since $\bar{A}$ may be put in the form $A \prec x$, it follows that $\bar{\bar{A}}$ may be put in the form $A \overline{\prec}\, x$, so that by the principles of contradiction and excluded middle, A may be put in the form $A \overline{\prec}\, x$. On the other hand, to say $S \overline{\prec}\, \bar{P}$ is the same as to say $(S \prec \bar{x}) \prec (P \overline{\prec}\, x)$, whatever x may be; for

$$(S \overline{\prec}\, \bar{P}) \prec \{(S \prec \bar{x}) \prec (P \overline{\prec}\, x)\}$$

is the principle of *Ferison*, a valid syllogism of the third figure; and if for x we put $\bar{S}$, we have

$$(S \prec \bar{\bar{S}}) \prec (P \overline{\prec}\, \bar{S}),$$

which is the same as to say that $P \overline{\prec}\, \bar{S}$ is true if the principle of contradiction is true. So that it follows that $P \overline{\prec}\, \bar{S}$ if $S \overline{\prec}\, \bar{P}$ from the principle of contradiction. Comparing

$$S \prec P \quad \text{or} \quad (S \overline{\prec}\, x) \prec (P \overline{\prec}\, x)$$

with

$$S \overline{\prec}\, \bar{P} \quad \text{or} \quad (S \prec \bar{x}) \prec (P \overline{\prec}\, x),$$

we see that they differ by a modification of the subject. Denoting this by a short curve over the subject, we may write $\breve{S} \prec P$ for $S \overline{\prec}\, \bar{P}$. We see then that while for A we may write $A \overline{\prec}\, x$, where x is anything whatever, so for $\breve{A}$ we may write $A \prec \bar{x}$. If we attach a similar modification to the predicate also, we have

$$\breve{S} \prec \breve{P} \quad \text{or} \quad (S \prec \bar{x}) \prec (P \prec \bar{x}),$$

which is the same as to say that you can find an S which is any P you please. We thus have

$$(\mathrm{S} \prec \mathrm{P}) \prec (\breve{\mathrm{P}} \prec \breve{\mathrm{S}}), \tag{24}$$

a formula of contraposition, similar to (16).

It is obvious that

$$(\breve{\mathrm{S}} \prec \mathrm{P}) \prec (\breve{\mathrm{P}} \prec \mathrm{S}); \tag{25}$$

for, negating both propositions, this becomes, by (16),

$$(\mathrm{P} \prec \bar{\mathrm{S}}) \prec (\mathrm{S} \prec \bar{\mathrm{P}}),$$

which is (19). The inference justified by (25) is called the conversion of I. From (25) we infer

$$\breve{\breve{x}} \prec x, \tag{26}$$

which may be called the principle of particularity. This is obviously true, because the modification of particularity only consists in changing $(\mathrm{A} \overline{\prec} x)$ to $(\mathrm{A} \prec \bar{x})$, which is the same as negating the copula and predicate, and a repetition of this will evidently give the first expression again. For the same reason we have

$$x \prec \breve{\breve{x}}, \tag{27}$$

which may be called the principle of individuality. This gives

$$(\mathrm{S} \prec \breve{\mathrm{P}}) \prec (\mathrm{P} \prec \breve{\mathrm{S}}), \tag{28}$$

and (26) and (27) together give

$$(\breve{\mathrm{S}} \prec \breve{\mathrm{P}}) \prec (\mathrm{P} \prec \mathrm{S}). \tag{29}$$

It is doubtful whether the proposition $\mathrm{S} \prec \breve{\mathrm{P}}$ ought to be interpreted as signifying that S and P are one sole individual, or that there is something besides S and P. I here leave this branch of the subject in an unfinished state.

Corresponding to the formulæ which we have obtained by the principle (2) are an equal number obtained by the following principle:

(2′) The inference

$$x$$
$$\therefore \text{ Either } y \text{ or } z$$

has the same validity as

$$x \overline{\prec} y$$
$$\therefore z.$$

From (1) we have

$$(x \overline{\prec} y) \prec (x \overline{\prec} y),$$

whence, by (2),

(4′)
$$x$$
$$\therefore \text{ Either } (x \prec y) \text{ or } y.$$

This gives

$$x$$
$$\therefore \text{ Either } x \mathrel{\overline{\prec}} y \text{ or } y \mathrel{\overline{\prec}} z \text{ or } z.$$

Then, by (2),

(5′)
$$x \mathrel{\overline{\prec}} z$$
$$\therefore x \mathrel{\overline{\prec}} y \text{ or } y \mathrel{\overline{\prec}} z,$$

which is the canonical form of dialogism. The minor indirect dialogism is

(8′)
$$x \mathrel{\overline{\prec}} (\mathrm{M} \mathrel{\overline{\prec}} \mathrm{P})$$
$$\therefore \text{ Either } x \mathrel{\overline{\prec}} (\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}) \text{ or } \mathrm{S} \mathrel{\overline{\prec}} \mathrm{M}.$$

The major indirect dialogism is

$$x \mathrel{\overline{\prec}} (\mathrm{S} \mathrel{\overline{\prec}} \mathrm{M})$$
$$\therefore \text{ Either } x \mathrel{\overline{\prec}} (\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}) \text{ or } \mathrm{M} \mathrel{\overline{\prec}} \mathrm{P}.$$

We have also

(12′)
$$(\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}) \mathrel{\overline{\prec}} x$$
$$\therefore \text{ Either } (\mathrm{S} \mathrel{\overline{\prec}} \mathrm{M}) \text{ or } (\mathrm{M} \mathrel{\overline{\prec}} \mathrm{P}) \mathrel{\overline{\prec}} x$$

and

(13′)
$$(\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}) \mathrel{\overline{\prec}} x$$
$$\therefore \text{ Either } (\mathrm{M} \mathrel{\overline{\prec}} \mathrm{P}) \text{ or } (\mathrm{S} \mathrel{\overline{\prec}} \mathrm{M}) \mathrel{\overline{\prec}} x.$$

We have A of the form $x \mathrel{\overline{\prec}} \bar{\mathrm{A}}$. And we have the inferences

$$\begin{array}{cccc} \mathrm{S} \mathrel{\overline{\prec}} \mathrm{P} & \mathrm{S} \mathrel{\overline{\prec}} \bar{\mathrm{P}} & \bar{\mathrm{S}} \mathrel{\overline{\prec}} \mathrm{P} & \bar{\mathrm{S}} \mathrel{\overline{\prec}} \bar{\mathrm{P}} \\ \therefore \bar{\mathrm{P}} \mathrel{\overline{\prec}} \bar{\mathrm{S}}. & \therefore \mathrm{P} \mathrel{\overline{\prec}} \bar{\mathrm{S}}. & \therefore \bar{\mathrm{P}} \mathrel{\overline{\prec}} \mathrm{S}. & \therefore \mathrm{P} \mathrel{\overline{\prec}} \mathrm{S}. \end{array}$$

CHAPTER II.—THE LOGIC OF NON-RELATIVE TERMS.

§ 1. *The Internal Multiplication and the Addition of Logic.*

We have seen that the inference

$$x \text{ and } y$$
$$\therefore z$$

is of the same validity with the inference

$$x$$
$$\therefore \text{ Either } \bar{y} \text{ or } z,$$

and the inference

$$x$$
$$\therefore \text{ Either } y \text{ or } z$$

with the inference

$$x \text{ and } \bar{y}$$
$$\therefore z.$$

In like manner,

$$x \prec y$$

is equivalent to

$$(\text{The possible}) \prec \text{Either } \bar{x} \text{ or } y,$$

and to

$$x \text{ which is } \bar{y} \prec (\text{The impossible}).$$

To express this algebraically, we need, in the first place, symbols for the two terms of second intention, the possible and the impossible. Let ∞ and 0 be the terms; then we have the definitions

$$x \prec \infty \qquad\qquad 0 \prec x \tag{1}$$

whatever x may be.*

We need also two operations which may be called non-relative addition and multiplication. They are defined as follows: †

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* The symbol 0 is used by Boole; the symbol ∞ replaces his 1, according to a suggestion in my *Logic of Relatives*, 1870.

† These forms of definition are original. The algebra of non-relative terms was given by Boole (*Mathematical Analysis of Logic*, 1847). Boole's addition was not the same as that in the text, for with him whatever was common to the two terms added was taken twice over in the sum. The operations in the text were given as complements of one another, and with appropriate symbols, by De Morgan (*On the Syllogism*, No. III., 1858, p. 185). For addition, sum, parts, he uses aggregation, aggregate, aggregants; for multiplication, product, factors, he uses composition, compound, components. Mr. Jevons (*Formal Logic*, 1864) — I regret that I can only speak of this work from having read it many years ago, and therefore cannot be sure of doing it full justice — improved the algebra of Boole by substituting De Morgan's aggregation for Boole's addition. The present writer, not having seen either De Morgan's or Jevons's writings on the subject, again recommended the same change (*On an Improvement in Boole's Calculus of Logic*, 1867), and showed the perfect balance existing between the two operations. In another paper, published in 1870, I introduced the sign of inclusion into the algebra.

In 1872, Robert Grassmann, brother of the author of the *Ausdehnungslehre*, published a work entitled '*Die Formenlehre oder Mathematik*,' the second book of which gives an algebra of logic identical with that of Jevons. The very notation is reproduced, except that the universe is denoted by T instead of U, and a term is negatived by drawing a line over it, as by Boole, instead of by taking a type from the other case, as Jevons does. Grassmann also uses a sign equivalent to my $\prec$. In his third book, he has other matter which he might have derived from my paper of 1870. Grassmann's treatment of the subject presents inequalities of strength; and most of his results had been anticipated. Professor Schröder, of Karlsruhe, in the spring of 1877, produced his *Operationskreis des Logikkalkuls*. He had seen the works of Boole and Grassmann, but not those of De Morgan, Jevons, and me. He gives a fine development of the algebra, adopting the addition of Jevons, and he exhibits the balance between $+$ and $\times$ by printing the theorems in parallel columns, thus imitating a practice of the geometricians. Schröder gives an original, interesting, and commodious method of working with the algebra. Later in the same year, Mr. Hugh McColl, apparently having known nothing of logical algebra except from a jejune account of Boole's work in *Bain's Logic*, published several papers on a *Calculus of Equivalent Statements*, the basis of which is nothing but the Boolian algebra, with Jevons's addition and a sign of inclusion. Mr. McColl adds an exceedingly ingenious application of this algebra to the transformation of definite integrals.

If $a \prec x$ and $b \prec x$, then $a + b \prec x$;	If $x \prec a$ and $x \prec b$, then $x \prec a \times b$;	(2)
and conversely, if $a + b \prec x$, then $a \prec x$ and $b \prec x$.	and conversely, if $x \prec a \times b$, then $x \prec a$ and $x \prec b$.	(3)

From these definitions we at once deduce the following formulæ:—

A. $\quad a \prec a + b \qquad a \times b \prec a$ (Peirce, 1870)* $\qquad$ (4)
$\quad b \prec a + b \qquad a \times b \prec b.$

These are proved by substituting $a + b$ and $a \times b$ for x in (3).

B. $\quad x = x + x \qquad x \times x = x$ (Jevons, 1864). $\qquad$ (5)

By substituting x for a and b in (2), we get

$$x + x \prec x \qquad x \prec x \times x;$$

and, by (4),

$$x \prec x + x \qquad x \times x \prec x.$$

C. $\quad a + b = b + a \qquad a \times b = b \times a$ (Boole, Jevons). $\qquad$ (6)

These formulæ are examples of the *commutative principle*. From (4) and (2),

$$b + a \prec a + b \qquad a \times b \prec b \times a$$

and interchanging a and b we get the reciprocal inclusion implied in (6).

D. $\quad (a + b) + c = a + (b + c) \qquad a \times (b \times c) = (a \times b) \times c$ (Boole, Jevons). $\qquad$ (7)

These are cases of the *associative principle*. By (4), $c \prec b + c$ and $b \times c \prec c$; also $b + c \prec a + (b + c)$ and $a \times (b \times c) \prec b \times c$; so that $c \prec a + (b + c)$ and $a \times (b \times c) \prec c$. In the same way, $b \prec a + (b + c)$ and $a \times (b \times c) \prec b$, and, by (4), $a \prec a + (b + c)$ and $a \times (b \times c) \prec a$. Hence, by (2), $a + b \prec a + (b + c)$ and $a \times (b \times c) \prec a \times b$. And, again by (2), $(a + b) + c \prec a + (b + c)$ and $a \times (b \times c) \prec (a \times b) \times c$. In a similar way we should prove the converse propositions to these and so establish (7).

E. $\quad (a + b) \times c = (a \times c) + (b \times c) \qquad (a \times b) + c = (a + c) \times (b + c)$.† $\qquad$ (8)

These are cases of the *distributive principle*. They are easily proved by (4) and (2), but the proof is too tedious to give.

F. $\quad (a + b) + c = (a + c) + (b + c) \qquad (a \times b) \times c = (a \times c) \times (b \times c)$. $\qquad$ (9)

* *Logic of Relatives* (§ 4); gives $a \times b \prec a$. The other formulæ, equally obvious, I do not find anywhere.

† The first of these given by Boole for his addition, was retained by Jevons in changing the addition. The second was first given by me (1867).

These are other cases of the distributive principle. They are proved by (5), (6) and (7). These formulæ, which have hitherto escaped notice, are not without interest.

G. $a+(a\times b)=a \qquad a\times(a+b)=a$ (Grassmann, Schröder). (10)

By (4), $a \prec a+(a\times b) \qquad a\times(a+b)\prec a.$

Again, by (4), $(a\times b)\prec a$ and $a\prec a+b$; hence, by (2)

$$a+(a\times b)\prec a \qquad a\prec a\times(a+b).$$

H. $$(a+b\prec a)=(b\prec a\times b). \tag{11}$$

This proposition is a transformation of Schröder's two propositions 21, (p. 25), one of which was given by Grassmann. By (3)

$$(a+b\prec a)\prec(b\prec a) \qquad (b\prec a\times b)\prec(b\prec a).$$

Hence, since $\qquad b\prec b, \qquad a\prec a$

we have, by (2),

$$(a+b\prec a)\prec(b\prec a\times b) \qquad (b\prec a\times b)\prec(a+b\prec a).$$

I. $$\left.\begin{aligned}(a\prec b)\times(x\prec y)\prec(a+x\prec b+y)\\(a\prec b)\times(x\prec y)\prec(a\times x\prec b\times y)\end{aligned}\right\}\text{(Peirce, 1870).} \tag{12}$$

Readily proved from (2) and (4).

J. $$(a\prec b+x)\times(a\times x\prec b)=(a\prec b). \tag{13}$$

This is a generalization of a theorem by Grassmann. In stating it, he erroneously unites the first two propositions by $+$ instead of $\times$. By (12), (5), and (8),

$$\begin{aligned}(a\prec b+x)\prec\{a\prec(a\times b)+(a\times x)\}\\(a\times x\prec b)\prec\{(a+b)\times(x+b)\prec b\}.\end{aligned}$$

But by (4)

$$a\prec a+b \qquad a\times b\prec b.$$

Hence, by (2), it is doubly proved that

$$(a\prec b+x)\times(a\times x\prec b)\prec(a\prec b).$$

The demonstration of the converse is obvious.

We have immediately, from (2) and (3),

K. $(a+b\prec c)=(a\prec c)\times(b\prec c) \quad (c\prec a\times b)=(c\prec a)\times(c\prec b)$ (14)

L. $$\begin{aligned}(c\prec a+b)&=\Sigma\,\{(p\prec a)\times(q\prec b)\}\quad\text{where } p+q=c\\(a\times b\prec c)&=\Sigma\,\{(a\prec p)\times(b\prec q)\}\quad\text{where } c=p\times q.\end{aligned} \tag{15}$$

The propositions (15) are new. By (12)

$$\begin{aligned}\{(p\prec a)\times(q\prec b)\}\prec(c\prec a+b)\quad\text{where } p+q=c\\\{(a\prec p)\times(b\prec q)\}\prec(a\times b\prec c)\quad\text{where } c=p\times q.\end{aligned}$$

And, since these are true for any set of values of p and q, we have by (2)

$$\Sigma\{(p \prec a)\times(q \prec b)\} \prec (c \prec a+b), \text{ where } p+q=c.$$
$$\Sigma\{(a \prec p)\times(b \prec q)\} \prec (a\times b \prec c), \text{ where } c=p\times q.$$

By (4) and (8), we have

$$(c \prec a+b) \prec \{(a\times c)+(b\times c)=c\}$$
$$(a\times b \prec c) \prec \{(c+a)\times(c+b)=c\}.$$

Hence, putting

$$a\times c=p \qquad b\times c=q, \qquad \text{where } p+q=c$$
$$a+c=p \qquad b+c=q, \qquad \text{where } p\times q=c,$$

we have

$$(c \prec a+b) \prec (p \prec a)\times(q \prec b), \text{ where } p+q=c$$
$$(a\times b \prec c) \prec (a \prec p)\times(b \prec q), \text{ where } c=p\times q,$$

whence, by (4)

$$(c \prec a+b) \prec \Sigma\{(p \prec a)\times(q \prec b)\} \text{ where } p+q=c$$
$$(a\times b \prec c) \prec \Sigma\{(a \prec p)\times(b \prec q)\} \text{ where } c=p\times q.$$

A formula analogous to (15) will be found below, (35).

From (1) and (2) and (4) we have

$$x+0=x \qquad x=x\times\infty. \tag{16}$$

From (1) and (4),

$$x+\infty=\infty \qquad 0=x\times 0. \tag{17}$$

The definition of the negative has as we have seen three clauses: first, that $\bar{a}$ is of the form $a \prec x$; second, $a \prec \bar{\bar{a}}$; third, $\bar{\bar{a}} \prec a$.

From the first we have that if

$$\begin{array}{cc} c & a \\ & \therefore\ b \end{array}$$

is valid, then

$$\begin{array}{cc} c & \bar{b} \\ & \therefore\ \bar{a} \end{array}$$

is valid. Or

$$(c\times a \prec b) \prec (c\times\bar{b} \prec \bar{a}). \tag{18}$$

Also, that if

$$\begin{array}{c} b \\ \therefore \text{ Either } c \text{ or } a \end{array}$$

is valid, then

$$\begin{array}{c} \bar{a} \\ \therefore \text{ Either } c \text{ or } \bar{b} \end{array}$$

is valid; or

$$(b \prec c+a) \prec (\bar{a} \prec c+\bar{b}). \tag{19}$$

Combining (18) and (19), we have

$$(a\times b \prec c+d) \prec (a\times\bar{d} \prec c+\bar{b}). \tag{20}$$

By the principles of contradiction and excluded middle, this gives

$$(a \times \bar{d} \prec c + \bar{b}) \prec (a \times b \prec c + d). \tag{21}$$

Thus the formula

$$(a \times b \prec c + d) = (a \times \bar{d} \prec c + \bar{b}) \tag{22}$$

embodies the essence of the negative.

If in (22) we put, first, $a = d \quad b = c = 0$, and then $a = d = \infty \quad b = c$, we have from the formula of identity

$$a \times \bar{a} = 0 \qquad a + \bar{a} = \infty. \tag{23}$$

We have

$$p = (p \times x) + (p \times \bar{x}) \qquad p = (p + x) \times (p + \bar{x}) \tag{24}$$

by the distributive principle and (23). If we write

$$i = p + (a \times \bar{x}) \qquad j = p + (b \times x) \qquad k = p \times (c + x) \qquad l = p \times (d + \bar{x}),$$

we equally have

$$p = (i \times x) + (j \times \bar{x}) \qquad p = (l + x) \times (k + \bar{x}). \tag{25}$$

Now p may be a function of x, and such values may perhaps be assigned to a, b, c, d, that i, j, k, l, shall be free from x. It is obvious that if the function results from any complication of the operations $+$ and $\times$, this is the case. Supposing, then, i, j, k, l, to be constant, we have, putting successively, 1, and 0, for x.

$$\phi \infty = i = k$$
$$\phi 0 = j = l$$

so that

$$\phi x = (\phi \infty \times x) + (\phi 0 \times \bar{x}) \qquad \phi x = (\phi 0 + x) \times (\phi \infty + \bar{x}). \tag{26}$$

The first of these formulæ was given by Boole for his addition. I showed (1867) that both hold for the modified addition. These formulæ are real analogues of mathematical developments; but practically they are not convenient. Their connection suggests the general formula

$$(a + x) \times (b + \bar{x}) = (a \times \bar{x}) + (b \times x) \tag{27}$$

a formula of frequent utility.

The distributive principle and (3) applied to (26) give

$$\phi 0 \times \phi \infty \prec \phi x \qquad \phi x \prec \phi \infty + \phi 0. \tag{28}$$

Hence

$$(\phi x = 0) \prec (\phi 0 \times \phi \infty = 0) \qquad (\phi x = \infty) \prec (\phi 0 + \phi \infty = \infty). \tag{29}$$

Boole gave the former, and I (1867) the latter. These formulæ are not convenient for elimination.

The following formulæ (probably given by De Morgan) are of great importance: —

$$\overline{a\times b} = \bar a + \bar b \qquad\qquad \overline{a+b} = \bar a \times \bar b. \tag{30}$$

By (23)

$$(a\times b)\times(\overline{a\times b}) \prec 0 \qquad\qquad \infty \prec (a+b)+(\overline{a+b}),$$

whence by (22) and the associative principle

$$b\times(\overline{a\times b}) \prec \bar a \qquad\qquad \bar a \prec b+(\overline{a+b})$$

$$\overline{a\times b} \prec \bar a+\bar b \qquad\qquad \bar a\times\bar b \prec \overline{a+b}.$$

By (4) and (22)

$$\bar a \prec \overline{a\times b} \qquad\qquad \overline{a+b} \prec \bar a$$

$$\bar b \prec \overline{a\times b} \qquad\qquad \overline{a+b} \prec \bar b,$$

whence by (2)

$$\bar a+\bar b \prec \overline{a+b} \qquad\qquad \overline{a+b} \prec \bar a\times\bar b.$$

The application of (22) gives from (11)

$$(b \mathrel{\overline{\prec}} a\times b) = (a+b \mathrel{\overline{\prec}} a); \tag{31}$$

from (12)

$$\left.\begin{array}{l}(a+x \mathrel{\overline{\prec}} b+y) \prec (a \mathrel{\overline{\prec}} b)+(x \mathrel{\overline{\prec}} y)\\ (a\times x \mathrel{\overline{\prec}} b\times y) \prec (a \mathrel{\overline{\prec}} b)+(x \mathrel{\overline{\prec}} y);\end{array}\right. \tag{32}$$

from (13)

$$(a \mathrel{\overline{\prec}} b) = (a \mathrel{\overline{\prec}} b+x)+(a\times x \mathrel{\overline{\prec}} b); \tag{33}$$

from (14)

$$(a+b \mathrel{\overline{\prec}} c)=(a \mathrel{\overline{\prec}} c)+(b \mathrel{\overline{\prec}} c) \qquad (c \mathrel{\overline{\prec}} a\times b)\prec(c \mathrel{\overline{\prec}} a)+(c \mathrel{\overline{\prec}} b); \tag{34}$$

from (15)

$$\left.\begin{array}{l}(c \mathrel{\overline{\prec}} a+b) = \Pi\,\{(p \mathrel{\overline{\prec}} a)+(q \mathrel{\overline{\prec}} b)\}\text{ where } p+q=c\\ (a\times b \mathrel{\overline{\prec}} c) = \Pi\,\{(a \mathrel{\overline{\prec}} p)+(b \mathrel{\overline{\prec}} q)\}\text{ where } p\times q=c;\end{array}\right. \tag{35}$$

from (22)

$$(a\times b \mathrel{\overline{\prec}} c+d) = (a\times \bar d \mathrel{\overline{\prec}} c+\bar b). \tag{36}$$

§ 2. *The Resolution of Problems in Non-relative Logic.*

Four different algebraic methods of solving problems in the logic of non-relative terms have already been proposed by Boole, Jevons, Schröder, and McColl. I propose here a fifth method which perhaps is simpler and certainly is more natural than any of the others. It involves the following processes:

First Process. Express all the premises with the copulas $\prec$ and $\overline{\prec}$, remembering that A = B is the same as A $\prec$ B and B $\prec$ A.

Second Process. Separate every predicate into as many factors and every subject into as many aggregant terms as is possible without increasing the number of different letters used in any subject or predicate.

An expression might be separated into such factors or aggregants (let us term them *prime* factors and *ultimate* aggregants) by one or other of these formulæ:

$$\phi x = (\phi\infty \times x) + (\phi 0 \times \bar{x})$$
$$\phi x = (\phi\infty + \bar{x}) \times (\phi 0 + x).$$

But the easiest method is this. To separate an expression into its $\left\{\begin{matrix}\text{ultimate aggregants}\\ \text{prime factors}\end{matrix}\right\}$ take any $\left\{\begin{matrix}\text{product}\\ \text{sum}\end{matrix}\right\}$ of all the different letters of the expression, each taken either positively or negatively (that is, with a dash over it). By means of the fundamental formulæ

$$\mathrm{X} \times \mathrm{Y} \prec \mathrm{Y} \prec \mathrm{Y} + \mathrm{Z},$$

examine whether the $\left\{\begin{matrix}\text{product}\\ \text{sum}\end{matrix}\right\}$ taken is a $\left\{\begin{matrix}\text{subject}\\ \text{predicate}\end{matrix}\right\}$ of every $\left\{\begin{matrix}\text{factor}\\ \text{aggregant}\end{matrix}\right\}$ of the given expression. If so, it is a $\left\{\begin{matrix}\text{ultimate aggregant}\\ \text{prime factor}\end{matrix}\right\}$ of that expression; otherwise not. Proceed in this way until as many $\left\{\begin{matrix}\text{ultimate aggregants}\\ \text{prime factors}\end{matrix}\right\}$ have been found as the expression possesses. This number is found in the case of a $\left\{\begin{matrix}\text{product of sums}\\ \text{sum of products}\end{matrix}\right\}$ of letters, as follows. Let m be the number of *different* letters in the expression (a letter and its negative not being considered different); let n be the total number of letters whether the same or different, and let p be the number of $\left\{\begin{matrix}\text{factors}\\ \text{terms}\end{matrix}\right\}$. Then the number of $\left\{\begin{matrix}\text{ultimate aggregants}\\ \text{prime factors}\end{matrix}\right\}$ is

$$2^m + n - mp - p.$$

For example, let it be required to separate $x + (y \times z)$ into its prime factors. Here $m = 3$, $n = 3$, $p = 2$. Hence the number of factors is three. Trying $x + y + z$, we have

$$x \prec x + y + z \qquad\qquad y \times z \prec x + y + z,$$

so that this is a factor. Trying $x + y + \bar{z}$, we have

$$x \prec x + y + \bar{z} \qquad\qquad y \times z \prec x + y + \bar{z},$$

so that this is also a factor. It is, also, obvious that $x + \bar{y} + z$ is the third factor. Accordingly,

$$x + (y \times z) = (x + y + z) \times (x + y + \bar{z}) \times (x + \bar{y} + z).$$

Again, let us develop the expression

$$(\bar{a} + b + c) \times (a + \bar{b} + \bar{c}) \times (a + b + c).$$

Here $m = 3$, $n = 9$, $p = 3$; so that the number of ultimate aggregants is five.

Of the eight possible products of three letters, then, only three are excluded, namely: $(a \times \bar{b} \times \bar{c})$, $(\bar{a} \times b \times c)$ and $(\bar{a} \times \bar{b} \times \bar{c})$. We have, then,

$$(\bar{a} + b + c) \times (a + \bar{b} + \bar{c}) \times (a + b + c) =$$
$$(a \times b \times c) + (a \times b \times \bar{c}) + (a \times \bar{b} \times c) + (\bar{a} \times b \times \bar{c}) + (\bar{a} \times \bar{b} \times c).$$

Third Process. Separate all complex propositions into simple ones by means of the following formulæ from the definitions of $+$ and $\times$:

$$\begin{aligned}
(\mathrm{X} + \mathrm{Y} \prec \mathrm{Z}) &= (\mathrm{X} \prec \mathrm{Z}) \times (\mathrm{Y} \prec \mathrm{Z}) \\
(\mathrm{X} \prec \mathrm{Y} \times \mathrm{Z}) &= (\mathrm{X} \prec \mathrm{Y}) \times (\mathrm{X} \prec \mathrm{Z}) \\
(\mathrm{X} + \mathrm{Y} \overline{\prec} \mathrm{Z}) &= (\mathrm{X} \overline{\prec} \mathrm{Z}) + (\mathrm{Y} \overline{\prec} \mathrm{Z}) \\
(\mathrm{X} \overline{\prec} \mathrm{Y} \times \mathrm{Z}) &= (\mathrm{X} \overline{\prec} \mathrm{Y}) + (\mathrm{X} \overline{\prec} \mathrm{Z}).
\end{aligned}$$

In practice, the first three operations will generally be performed off-hand in writing down the premises.

Fourth Process. If we have given two propositions, one of one of the forms

$$a \prec b + x \qquad\qquad a \times \bar{x} \prec b,$$

and the other of one of the forms

$$c \prec d + \bar{x} \qquad\qquad c \times x \prec d,$$

we may, by the transitiveness of the copula, eliminate x, and so obtain

$$a \times c \prec b + d.$$

Fifth Process. We may transpose any term from subject to predicate or the reverse, by changing it from positive to negative or the reverse, and at the same time its mode of connection from addition to multiplication or the reverse. Thus,

$$(x \times y \prec z) = (x \prec \bar{y} + z).$$

We may, in this way, obtain all the subjects and predicates of any letter; or we may bring all the letters into the subject, leaving the predicate 0, or all into the predicate, leaving the subject ∞.

Sixth Process. Any number of propositions having a common $\left\{\begin{matrix}\text{subject}\\ \text{predicate}\end{matrix}\right\}$ are, taken together, equivalent to their $\left\{\begin{matrix}\text{product}\\ \text{sum}\end{matrix}\right\}$.

As an example of this method, we may consider a well-known problem given by Boole. The data are

$$\begin{gathered}
\bar{x} \times \bar{z} \prec v \times (y \times \bar{w} + \bar{y} \times w) \\
\bar{v} \times x \times w \prec (y \times z) + (\bar{y} \times \bar{z}) \\
(x \times y) + (v \times x \times \bar{y}) = (z \times \bar{w}) + (\bar{z} \times w).
\end{gathered}$$

The quæsita are: first, to find those predicates of x which involve only y, z, and w; second, to find any relations which may be implied between y, z, w; third, to find the predicates of y; fourth, to find any relation which may be implied between x, z, and w. By the first three processes, mentally performed, we resolve the premises as follows: the first into

$$\begin{aligned}
&\bar{x} \times \bar{z} \prec v \\
&\bar{x} \times \bar{z} \prec y + w \\
&\bar{x} \times \bar{z} \prec \bar{y} + \bar{w};
\end{aligned}$$

the second into

$$\begin{aligned}
&\bar{v} \times x \times w \prec y + \bar{z} \\
&\bar{v} \times x \times w \prec \bar{y} + z;
\end{aligned}$$

the third into

$$\begin{aligned}
&x \times y \prec z + w \\
&x \times y \prec \bar{z} + \bar{w} \\
&v \times x \times \bar{y} \prec z + w \\
&v \times x \times \bar{y} \prec \bar{z} + \bar{w} \\
&z \times \bar{w} \prec x \\
&\bar{z} + w \prec v + y \\
&z + \bar{w} \prec x \\
&\bar{z} + w \prec v + y.
\end{aligned}$$

We must first eliminate v, about which we want to know nothing. We have, on the one hand, the propositions

$$\begin{aligned}
&v \times x \times \bar{y} \prec z + w \\
&v \times x \times \bar{y} \prec \bar{z} + \bar{w};
\end{aligned}$$

and, on the other, the propositions

$$\begin{aligned}
&\bar{x} \times \bar{z} \prec v \\
&\bar{v} \times x \times w \prec y + \bar{z} \\
&\bar{v} \times x \times w \prec \bar{y} + z \\
&z \times \bar{w} \prec v + y \\
&\bar{z} \times w \prec v + y.
\end{aligned}$$

The conclusions from these propositions are obtained by taking one from each set, multiplying their subjects, adding their predicates, and omitting v. The result will be a merely empty proposition if the same letter in the same quality as to being positive or negative be found in the subject and in the predicate, or if it be found twice with opposite qualities either in the subject or in the predicate. Thus, it will be useless to combine the proposition $v \times x \times \bar{y} \prec z + w$ with any which contains $\bar{x}$, y, z, or w, in the subject. But all of the second set do this, so that nothing can be concluded from this proposition. So it will be useless to combine $v \times x \times \bar{y} \prec \bar{z} + \bar{w}$ with any which contains $\bar{x}, y, \bar{z}, \bar{w}$ in the subject, or z in the predicate. This excludes every proposition of the second set except $\bar{v} \times x \times w \prec y + \bar{z}$, which, combined with the proposition under discussion, gives

$$x \times w \prec y + \bar{z} + \bar{w}$$

or

$$x \times w \prec y + \bar{z},$$

which is therefore to be used in place of all the premises containing v.

One of the other propositions, namely, $\bar{x} \times \bar{z} \prec \bar{y} + \bar{w}$ is obviously contained in another, namely: $\bar{z} \times w \prec x$. Rejecting it, our premises are reduced to six, namely:

$$\begin{aligned} &\bar{x} \times \bar{z} \prec y + w \\ &x \times y \prec z + w \\ &x \times y \prec \bar{z} + \bar{w} \\ &z \times \bar{w} \prec x \\ &\bar{z} \times w \prec x \\ &x \times w \prec y + \bar{z}. \end{aligned}$$

The second, third, and sixth of these give the predicates of x. Their product is

$$x \prec (\bar{y} + z + w) \times (\bar{y} + \bar{z} + \bar{w}) \times (y + \bar{z} + \bar{w})$$

or

$$x \prec y \times z \times \bar{w} + y \times \bar{z} \times w + \bar{y} \times z \times \bar{w} + \bar{y} \times \bar{z} \times w + \bar{y} \times \bar{z} \times \bar{w}$$

or

$$x \prec z \times \bar{w} + \bar{z} \times w + \bar{y} \times \bar{z} \times \bar{w}.$$

To find whether any relation between y, z, and w can be obtained by the elimination of x, we find the subjects of x by combining the first, fourth, and fifth premises. Thus we find

$$\bar{y} \times \bar{z} \times \bar{w} + z \times \bar{w} + \bar{z} \times w \prec x.$$

It is obvious that the conclusion from the last two propositions is a merely identical proposition, and therefore no independent relation is implied between y, z, and w.

To find the predicates of y we combine the second and third propositions. This gives

$$y \prec (\bar{x} + z + w) \times (\bar{x} + \bar{z} + \bar{w})$$

or

$$y \prec x \times z \times \bar{w} + x \times \bar{z} \times w + \bar{x}.$$

Two relations between x, z, and w are given in the premises, namely: $z \times \bar{w} \prec x$ and $\bar{z} \times w \prec x$. To find whether any other is implied, we eliminate y between the above proposition and the first and sixth premises. This gives

$$\begin{aligned} &\bar{x} \times \bar{z} \prec x \times z \times \bar{w} + w + \bar{x} \\ &x \times w \prec x \times z \times \bar{w} + \bar{x} + \bar{z}. \end{aligned}$$

The first conclusion is empty. The second is equivalent to $x \times w \prec \bar{z}$, which is a third relation between x, z, and w.

Everything implied in the premises in regard to the relations of x, y, z, w may be summed up in the proposition

$$\infty \prec x + z \times w + y \times \bar{z} \times \bar{w}.$$

CHAPTER III. — THE LOGIC OF RELATIVES.

§ 1. *Individual and Simple Terms.*

Just as we had to begin the study of Logical Addition and Multiplication by considering ∞ and 0, terms which might have been introduced under the Algebra of the Copula, being defined in terms of the copula only, without the use of $+$ or $\times$, but which had not been there introduced, because they had no application there, so we have to begin the study of relatives by considering the doctrine of individuals and simples, — a doctrine which makes use only of the conceptions of non-relative logic, but which is wholly without use in that part of the subject, while it is the very foundation of the conception of a relative, and the basis of the method of working with the algebra of relatives.

The germ of the correct theory of individuals and simples is to be found in Kant's *Critic of the Pure Reason, Appendix to the Transcendental Dialectic,* where he lays it down as a regulative principle, that, if

$$a \prec b \qquad\qquad b \mathrel{\overline{\prec}} a,$$

then it is always possible to find such a term x, that

$$a \prec x \qquad\qquad x \prec b$$
$$x \mathrel{\overline{\prec}} a \qquad\qquad b \mathrel{\overline{\prec}} x.$$

Kant's distinction of regulative and constitutive principles is unsound, but this *law of continuity,* as he calls it, must be accepted as a fact. The proof of it, which I have given elsewhere, depends on the continuity of space, time, and the intensities of the qualities which enter into the definition of any term. If, for instance, we say that Europe, Asia, Africa and North America are continents, but not all the continents, there remains over only South America. But we may distinguish between South America as it now exists and South America in former geological times; we may, therefore, take x as including Europe, Asia, Africa, North America, and South America as it exists now, and every x is a continent, but not every continent is x.

Just as in mathematics we speak of infinitesimals and infinites, which are fictitious limits of continuous quantity, and every statement involving these expressions has its interpretation in the doctrine of limits, so in logic we may define an *individual*, A, as such a term that

$$\mathrm{A} \;\overline{-\!\!<}\; 0,$$

but such that if

$$x < \mathrm{A}$$

then $$x -\!\!< 0.$$

And in the same way, we may define the *simple*, α, as such a term that

$$1 \;\overline{-\!\!<}\; \alpha,$$

but such that if

$$\alpha < x$$

then $$1 -\!\!< x.$$

The individual and the simple, as here defined, are ideal limits, and every statement about either is to be interpreted by the doctrine of limits.

Every term may be conceived as a limitless logical sum of individuals, or as a limitless logical product of simples; thus,

$$a = \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{A}_4 + \mathrm{A}_5 + \text{etc.}$$
$$\bar{a} = \bar{\mathrm{A}}_1 \times \bar{\mathrm{A}}_2 \times \bar{\mathrm{A}}_3 \times \bar{\mathrm{A}}_4 \times \bar{\mathrm{A}}_5 \times \text{etc.}$$

It will be seen that a simple is the negative of an individual.

§ 2. *Relatives.*

A *relative* is a term whose definition describes what sort of a system of objects that is whose first member (which is termed the *relate*) is denoted by the term; and names for the other members of the system (which are termed the *correlates*) are usually appended to limit the denotation still further. In these systems the order of the members is essential; so that (A, B, C) and (A, C, B) are different systems. As an example of a relative, take 'buyer of — for — from'; we may append to this three correlates, thus, 'buyer of every horse of a certain description in the market for a good price from its owner.'

A relative of only one correlate, so that the system it supposes is a pair, may be called a *dual* relative; a relative of more than one correlate may be called *plural*. A non-relative term may be called a term of *singular reference*.

Every relative, like every term of singular reference, is general; its definition describes a system in general terms; and, as general, it may be conceived either as a logical sum of individual relatives, or as a logical product of simple relatives.* An individual relative refers to a system all the members of which are individual. The expressions

$$(A : B) \qquad (A : B : C)$$

may denote individual relatives. Taking dual individual relatives, for instance, we may arrange them all in an infinite block, thus,

$$\begin{array}{cccccc} A : A & A : B & A : C & A : D & A : E & \text{etc.} \\ B : A & B : B & B : C & B : D & B : E & \text{etc.} \\ C : A & C : B & C : C & C : D & C : E & \text{etc.} \\ D : A & D : B & D : C & D : D & D : E & \text{etc.} \\ E : A & E : B & E : C & E : D & E : E & \text{etc.} \\ \text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} & \end{array}$$

In the same way, triple individual relatives may be arranged in a cube, and so forth. The logical sum of all the relatives in this infinite block will be the relative universe, ∞, where

$$x \prec \infty,$$

whatever dual relative x may be. It is needless to distinguish the dual universe, the triple universe, etc., because, by adding a perfectly indefinite additional member to the system, a dual relative may be converted into a triple relative, etc. Thus, for *lover of a woman*, we may write *lover of a woman coexisting with anything.* In the same way, a term of single reference is equivalent to a relative with an indefinite correlate; thus, *woman* is equivalent to *woman coexisting with anything.* Thus, we shall have

$$A = A : A + A : B + A : C + A : D + A : E + \text{etc.}$$
$$A : B = A : B : A + A : B : B + A : B : C + A : B : D + \text{etc.}$$

From the definition of a simple term given in the last section, it follows that every simple relative is the negative of an individual term. But while in non-relative logic negation only divides the universe into two parts, in relative logic the same operation divides the universe into 2^n parts, where n is the number of objects in the system which the relative supposes; thus,

$$\infty = A + \bar{A}$$
$$\infty = A : B + \bar{A} : B + A : \bar{B} + \bar{A} : \bar{B}$$

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* In my *Logic of Relatives*, 1870, I have used this expression to designate what I now call *dual relatives.*

$$\infty = (\mathrm{A : B : C}) + (\bar{\mathrm{A}} : \mathrm{B : C}) + (\mathrm{A} : \bar{\mathrm{B}} : \mathrm{C}) + (\mathrm{A : B} : \bar{\mathrm{C}})$$
$$+ (\bar{\mathrm{A}} : \bar{\mathrm{B}} : \bar{\mathrm{C}}) + (\mathrm{A} : \bar{\mathrm{B}} : \bar{\mathrm{C}}) + (\bar{\mathrm{A}} : \mathrm{B} : \bar{\mathrm{C}}) + (\bar{\mathrm{A}} : \bar{\mathrm{B}} : \mathrm{C}).$$

Here, we have

$$\mathrm{A = A : B + A} : \bar{\mathrm{B}}; \quad \bar{\mathrm{A}} = \bar{\mathrm{A}} : \mathrm{B} + \bar{\mathrm{A}} : \bar{\mathrm{B}};$$
$$\mathrm{A : B = A : B : C + A : B} : \bar{\mathrm{C}}; \quad \mathrm{A} : \bar{\mathrm{B}} = \mathrm{A} : \bar{\mathrm{B}} : \mathrm{C} + \mathrm{A} : \bar{\mathrm{B}} : \bar{\mathrm{C}};$$
$$\bar{\mathrm{A}} : \mathrm{B} = \bar{\mathrm{A}} : \mathrm{B : C} + \bar{\mathrm{A}} : \mathrm{B} : \bar{\mathrm{C}}; \quad \bar{\mathrm{A}} : \bar{\mathrm{B}} = \bar{\mathrm{A}} : \bar{\mathrm{B}} : \mathrm{C} + \bar{\mathrm{A}} : \bar{\mathrm{B}} : \bar{\mathrm{C}}.$$

It will be seen that a term which is individual when considered as n-fold is not so when considered as more than n-fold; but an n-fold term when made $(m+n)$-fold, is individual as to n members of the system, and indefinite as to m members.

Instead of considering the system of a relative as consisting of non-relative individuals, we may conceive of it as consisting of relative individuals. Thus, since

$$\mathrm{A = A : A + A : B + A : C + A : D + etc.},$$

we have

$$\mathrm{A : B = (A : A) : B + (A : B) : B + (A : C) : B + (A : D) : B + etc.}$$

But

$$\mathrm{B = B : A + B : B + B : C + B : D + etc.};$$

so that

$$\mathrm{A : B = A : (B : A) + A : (B : B) + A : (B : C) + A : (B : D) + etc.}$$

Here we have evidently

$$\mathrm{(A : C) : B = A : (B : C)}.$$

In the same way we find

$$\mathrm{(A : D) : (B : C) = (A : C) : (B : D)}$$
$$\mathrm{= A : [(B : D) : C] = A : [B : (C : D)]}$$
$$\mathrm{= [A : (C : D)] : B = [(A : D) : C] : B}.$$

§ 3. *Relatives connected by Transposition of Relate and Correlate.*

Connected with every dual relative, as

$$l = \Sigma\,(\mathrm{A : B}) = \Pi\,(\alpha : \beta),$$

is another which is called its *converse*,

$$k\text{-}l = \Sigma\,(\mathrm{B : A}) = \Pi\,(\beta : \alpha),$$

in which the relate and correlate are transposed. The converse, k, is itself a relative, being

$$k = \Sigma\,[(\mathrm{A : B}) : (\mathrm{B : A})];$$

that is, it is the first of any pair which embraces two individual dual relatives, each of which is the converse of the other. The converse of the converse is the relation itself, thus

$$k\text{-}k\text{-}l = l,$$

or say

$$kk = 1.$$

We have also

$$k\text{-}\bar{l} = \overline{k\text{-}l}$$
$$k\Sigma = \Sigma k$$
$$k\Pi = \Pi k.$$

In the case of triple relatives there are five transpositions possible. Thus, if

$$b = \Sigma\,[(\mathrm{A} : \mathrm{B}) : \mathrm{C}] = \Sigma\,[\mathrm{A} : (\mathrm{C} : \mathrm{B})],$$

we may write

$$\begin{aligned}
\mathrm{I}b &= \Sigma\,[(\mathrm{B} : \mathrm{A}) : \mathrm{C}] = \Sigma\,[\mathrm{B} : (\mathrm{C} : \mathrm{A})] \\
\mathrm{J}b &= \Sigma\,[\mathrm{A} : (\mathrm{B} : \mathrm{C})] = \Sigma\,[(\mathrm{A} : \mathrm{C}) : \mathrm{B}] \\
\mathrm{K}b &= \Sigma\,[\mathrm{C} : (\mathrm{A} : \mathrm{B})] = \Sigma\,[(\mathrm{C} : \mathrm{B}) : \mathrm{A}] \\
\mathrm{L}b &= \Sigma\,[(\mathrm{C} : \mathrm{A}) : \mathrm{B}] = \Sigma\,[\mathrm{C} : (\mathrm{B} : \mathrm{A})] \\
\mathrm{M}b &= \Sigma\,[\mathrm{B} : (\mathrm{A} : \mathrm{C})] = \Sigma\,[(\mathrm{B} : \mathrm{C}) : \mathrm{A}].
\end{aligned}$$

Here we have

$$\begin{aligned}
&\mathrm{LM} = \mathrm{ML} = 1 \\
&\mathrm{II} = \mathrm{JJ} = \mathrm{KK} = 1 \\
&\mathrm{IJ} = \mathrm{JK} = \mathrm{KI} = \mathrm{L} \\
&\mathrm{JI} = \mathrm{KJ} = \mathrm{IK} = \mathrm{M} \\
&\mathrm{IL} = \mathrm{MI} = \mathrm{J} = \mathrm{KM} = \mathrm{LK} \\
&\mathrm{JL} = \mathrm{MJ} = \mathrm{K} = \mathrm{IM} = \mathrm{LI} \\
&\mathrm{KL} = \mathrm{MK} = \mathrm{I} = \mathrm{JM} = \mathrm{LJ}.
\end{aligned}$$

If we write $a : b$ to express the operation of putting A in place of B in the original relative

$$b = \Sigma\,[(\mathrm{A} : \mathrm{B}) : \mathrm{C}] = \Sigma\,[\mathrm{A} : (\mathrm{C} : \mathrm{B})],$$

then we have

$$\begin{aligned}
\mathrm{I} &= a : b + b : a + c : c \\
\mathrm{J} &= a : a + b : c + c : b \\
\mathrm{K} &= a : c + b : b + c : a \\
\mathrm{L} &= a : b + b : c + c : a \\
\mathrm{M} &= a : c + b : a + c : b \\
1 &= a : a + b : b + c : c.
\end{aligned}$$

Then we have

$$\mathrm{I} + \mathrm{J} + \mathrm{K} = 1 + \mathrm{L} + \mathrm{M},$$

which does not imply

$$(\mathrm{I} + \mathrm{J} + \mathrm{K})\, l = (1 + \mathrm{L} + \mathrm{M})\, l.$$

In a similar way the n-fold relative will have $(n\,! - 1)$ transposition-functions.

§ 4. *Classification of Relatives.*

Individual relatives are of one or other of the two forms

$$\mathrm{A} : \mathrm{A} \qquad\qquad \mathrm{A} : \mathrm{B},$$

and simple relatives are negatives of one or other of these two forms.

The forms of general relatives are of infinite variety, but the following may be particularly noticed.

Relatives may be divided into those all whose individual aggregants are of the form A : A and those which contain individuals of the form A : B. The former may be called *concurrents,* the latter *opponents.* Concurrents express a mere agreement among objects. Such, for instance, is the relative '*man that is* —,' and a similar relative may be formed from any term of singular reference. We may denote such a relative by the symbol for the term of singular reference with a comma after it; thus $(m,)$ will denote 'man that is —' if (m) denotes 'man.' In the same way a comma affixed to an n-fold relative will convert it into an $(n+1)$-fold relative. Thus, (l) being 'lover of —,' $(l,)$ will be 'lover that is — of —.'

The negative of a concurrent relative will be one each of whose simple components is of the form $\overline{\mathrm{A} : \mathrm{A}}$, and the negative of an opponent relative will be one which has components of the form $\overline{\mathrm{A} : \mathrm{B}}$.

We may also divide relatives into those which contain individual aggregants of the form A : A and those which contain only aggregants of the form A : B. The former may be called *self-relatives,* the latter *alio-relatives.* We also have negatives of self-relatives and negatives of alio-relatives.

These different classes have the following relations. Every negative of a concurrent and every alio-relative is both an opponent and the negative of a self-relative. Every concurrent and every negative of an alio-relative is both a self-relative and the negative of an opponent. There is only one relative which is both a concurrent and the negative of an alio-relative; this is 'identical with —.' There is only one relative which is at once an alio-relative and the negative of a concurrent; this is the negative of the last, namely, 'other than —.' The following pairs of classes are mutually exclusive, and divide all relatives between them:

Alio-relatives and self-relatives,
Concurrents and opponents,
Negatives of alio-relatives and negatives of self-relatives,
Negatives of concurrents and negatives of opponents.

No relative can be at once either an alio-relative or the negative of a concurrent, and at the same time either a concurrent or the negative of an alio-relative.

We may append to the symbol of any relative a semicolon to convert it into an alio-relative of a higher order. Thus $(l;)$ will denote a 'lover of — that is not —.'

This completes the classification of dual relatives founded on the difference of the fundamental forms A : A and A : B. Similar considerations applied to triple relatives would give rise to a highly complicated development, inasmuch as here we have no less than five fundamental forms of individuals, namely,

$$(A:A):A \quad (A:A):B \quad (A:B):A \quad (B:A):A \quad (A:B):C.$$

The number of individual forms for the $(n+2)$-fold relative is

$$2+(2^n-1)\,.\,3+\frac{1}{2!}\left\{(3^n-1)-2\,(2^n-1)\right\}.\,4+\frac{1}{3!}\left\{(4^n-1)-3\,(3^n-1)\right.$$
$$\left.+3\,(2^n-1)\right\}.\,5+\frac{1}{4!}\left\{(5^n-1)-4\,(4^n-1)+6\,(3^n-1)-4\,(2^n-1)\right\}.\,6$$
$$+\frac{1}{5!}\left\{(6^n-1)-5\,(5^n-1)+10\,(4^n-1)-10\,(3^n-1)+5\,(2^n-1)\right\}.\,7+\text{etc.}$$

If this number be called fn, we have

$$\Delta^n f0 = f(n-1)$$
$$f0 = 1.$$

The form of calculation is

1					
2	1				
5	3	2			
15	10	7	5		
52	37	27	20	15	
203	151	114	87	67	52

where the diagonal line is copied number by number from the vertical line, as fast as the latter is computed.

Relatives may also be classified according to the general amount of filling up of the above-mentioned block, cube, etc. they present. In the first place, we have such relatives in whose block, cube, etc. every line in a certain direction in which there is a single individual is completely filled up. Such relatives may be called *complete in regard to* the relate, or first, second, third, etc. correlate. The dual relatives which are equivalent to terms of singular reference are complete as to their correlate.

A relative may be incomplete with reference to a certain correlate or to its relate, and yet every individual of the universe may in some way enter into that correlate or relate. Such a relative may be called *unlimited* in reference to the correlate or relate in question. Thus, the relative

$$A : A + A : B + C : C + C : D + E : E + E : F + G : G + G : H + \text{etc.}$$

is unlimited as to its correlate. The negative of an unlimited relative will be unlimited unless the relative has as an integrant a relative which is complete with regard to every other relate and correlate than that with reference to which the given relative is unlimited.

A totally unlimited relative is one which is unlimited in reference to the relate and all the correlates. A totally unlimited relative in which each letter enters only once into the relate and once into the correlate is termed a *substitution.*

Certain classes of relatives are characterized by the occurrence or non-occurrence of certain individual aggregants related in a definite way to others which occur. A set of individual dual relatives each of which has for its relate the correlate of the last, the last of all being considered as preceding the first of all, may be called a *cycle.* If there are n individuals in the cycle it may be called a cycle of the n^{th} order. For instance,

$$A : B \quad B : C \quad C : D \quad D : E \quad E : A$$

may be called the cycle of the fifth order. Now, if a certain relative be such that of any cycle of the n^{th} order of which it contains any m terms, it also contains the remaining $(n - m)$ terms, it may be called a cyclic relative of the n^{th} order and m^{th} degree. If. on the other hand, of any cycle of the n^{th} order of which it contains m terms the remaining $(n - m)$ are wanting, the relative may be called an anticyclic relative of the n^{th} order and m^{th} degree.

A cyclic relative of the first order and first degree contains all individual components of the form $A : A$. A cyclic relative of the second order and first degree is called an *equiparant* in opposition to a *disquiparant.*

A highly important class of relatives is that of *transitives;* that is to say, those which for every two individual terms of the forms $(A : B)$ and $(B : C)$ also possess a term of the form $(A : C)$.

§ 5. *The Composition of Relatives.*

Suppose two relatives either individual or simple, and having the relate or correlate of the first identical with the relate or correlate of the second or of its negative. This pair of relatives will then be of one or other of sixteen forms, viz.:

$$
\begin{array}{llll}
(A:B)\ (B:C) & (\overline{A:B})\ (B:C) & (A:B)\ (\overline{B:C}) & (\overline{A:B})\ (\overline{B:C}) \\
(A:B)\ (C:B) & (\overline{A:B})\ (C:B) & (A:B)\ (\overline{C:B}) & (\overline{A:B})\ (\overline{C:B}) \\
(B:A)\ (B:C) & (\overline{B:A})\ (B:C) & (B:A)\ (\overline{B:C}) & (\overline{B:A})\ (\overline{B:C}) \\
(B:A)\ (C:B) & (\overline{B:A})\ (C:B) & (B:A)\ (\overline{C:B}) & (\overline{B:A})\ (\overline{C:B}).
\end{array}
$$

Now we may conceive an operation upon any one of these sixteen pairs of relatives of such a nature that it will produce one or other of the four forms $(A:C)$, $(\overline{A:C})$, $(C:A)$, $(\overline{C:A})$. Thus, we shall have sixty-four operations in all.

We may symbolize them as follows: Let

$$A:B\ (|||)\ B:C = A:C\,;$$

that is, let $(|||)$ signify such an operation that this formula necessarily holds. The three lines in the sign of this operation are to refer respectively to the first relative operated upon, the second relative operated upon, and to the result. When either of these lines is replaced by a hyphen $(-)$, let the operation signified be such that the negative of the corresponding relative must be substituted in the above formula. Thus,

$$\overline{A:B}\ (-||)\ B:C = A:C.$$

In the same way, let a semicircle $(\smile)$ signify that the converse of the corresponding relative is to be taken. The hyphen and the semicircle may be used together. If, then, we write the symbol of a relative with a semicircle or curve over it to denote the converse of that relative, we shall have, for example,

$$\overset{\smile}{A:B}\ (\smile||)\ B:C = A:C.$$

Then any combination of the relatives a and e, in this order, is equivalent to others formed from this by making any of the following changes:

First. Putting a straight or curved mark over a and changing the first mark of the sign of operation in the corresponding way; that is,

for $\breve{a}$, from $|$ to $\smile$ or from $-$ to $\underline{\smile}$ or conversely,
for $\bar{a}$, from $|$ to $-$ or from $\smile$ to $\underline{\smile}$ or conversely,
for $\breve{\bar{a}}$, from $|$ to $\underline{\smile}$ or from $-$ to $\smile$ or conversely.

Second. Making similar simultaneous modifications of e and of the second mark.

Third. Changing the third mark from $|$ to $-$ or from $\smile$ to $\underline{\smile}$ or conversely, and simultaneously writing the mark of negation over the whole expression.

Fourth. Changing the third mark from | to $\smile$ or from – to $\underline{\smile}$ or conversely, and interchanging a and e and also the first and second marks.

We have thus far defined the effect of the sixty-four operations when certain members of the individual relatives operated upon are identical. When these members are not identical, we may suppose either that the operation ||| produces either the first or second relative or 0. We cannot suppose that it produces ∞ for a reason which will appear further on. Let us elect the formula

$$\mathrm{A} : \mathrm{B}\ (|||)\ \mathrm{C} : \mathrm{D} = 0.$$

The other excluded operations will be included in a certain manner, as we shall see below. From this formula, by means of the rules of equivalence, it will follow that all operations in whose symbol there is no hyphen in the third place will also give 0 in like circumstances, while all others will give $\bar{0}$ or ∞.

We have thus far only defined the effect of the sixty-four operations upon individual or simple terms. To extend the definitions to other cases, let us suppose first that the rules of equivalence are generally valid, and second, that

$$\text{If } a \prec b \text{ and } c \prec d \text{ then } a\ (|||)\ c \prec b\ (|||)\ d$$

or

$$(a \prec b) \times (c \prec d) \prec \{a\ (|||)\ c \prec b\ (|||)\ d\}.$$

Then, this rule will hold good in all operations in whose symbols the first and second places agree with the third in respect to having or not having hyphens. For operations, in whose symbols the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ mark disagrees with the third in this respect we must write $\left\{\begin{matrix} b \prec a \\ d \prec c \end{matrix}\right\}$ instead of $\left\{\begin{matrix} a \prec b \\ c \prec d \end{matrix}\right\}$ in this rule. Thus, the sixty-four operations are divisible into four classes according to which one of the four rules so produced they follow.

It now appears that only the hyphens and not the curved marks are of significance in reference to the rule which an operation follows. Let us accordingly reject all operations whose symbols contain curved marks, and there remain only eight. For these eight the following formulæ hold:

$$\begin{array}{ll}
\mathrm{A}:\mathrm{B}\ (|||)\ \mathrm{B}:\mathrm{C} = \mathrm{A}:\mathrm{C} & \mathrm{A}:\mathrm{B}\ (||-)\ \mathrm{B}:\mathrm{C} = \overline{\mathrm{A}:\mathrm{C}} \\
\overline{\mathrm{A}:\mathrm{B}}\ (-||)\ \mathrm{B}:\mathrm{C} = \mathrm{A}:\mathrm{C} & \overline{\mathrm{A}:\mathrm{B}}\ (-|-)\ \mathrm{B}:\mathrm{C} = \overline{\mathrm{A}:\mathrm{C}} \\
\mathrm{A}:\mathrm{B}\ (|-|)\ \overline{\mathrm{B}:\mathrm{C}} = \mathrm{A}:\mathrm{C} & \mathrm{A}:\mathrm{B}\ (|--)\ \overline{\mathrm{B}:\mathrm{C}} = \overline{\mathrm{A}:\mathrm{C}} \\
\overline{\mathrm{A}:\mathrm{B}}\ (--|)\ \overline{\mathrm{B}:\mathrm{C}} = \mathrm{A}:\mathrm{C} & \overline{\mathrm{A}:\mathrm{B}}\ (---)\ \overline{\mathrm{B}:\mathrm{C}} = \overline{\mathrm{A}:\mathrm{C}}
\end{array}$$

$$\mathrm{A:B}\ (\mid\mid\mid)\ \mathrm{C:D} = 0 \qquad \mathrm{A:B}\ (\mid\mid -)\ \mathrm{C:D} = \infty$$
$$\overline{\mathrm{A:B}}\ (-\mid\mid)\ \mathrm{C:D} = 0 \qquad \overline{\mathrm{A:B}}\ (-\mid -)\ \mathrm{C:D} = \infty$$
$$\mathrm{A:B}\ (\mid - \mid)\ \overline{\mathrm{C:D}} = 0 \qquad \mathrm{A:B}\ (\mid - -)\ \overline{\mathrm{C:D}} = \infty$$
$$\overline{\mathrm{A:B}}\ (- - \mid)\ \overline{\mathrm{C:D}} = 0 \qquad \overline{\mathrm{A:B}}\ (- - -)\ \overline{\mathrm{C:D}} = \infty$$

$$(a \prec b) \times (c \prec d) \prec \{a\ (\mid\mid\mid)\ c \prec b\ (\mid\mid\mid)\ d\}$$
$$(a \prec b) \times (c \prec d) \prec \{a\ (- - -)\ c \prec b\ (- - -)\ d\}$$

$$(b \prec a) \times (c \prec d) \prec \{a\ (-\mid\mid)\ c \prec b\ (-\mid\mid)\ d\}$$
$$(b \prec a) \times (c \prec d) \prec \{a\ (\mid - -)\ c \prec b\ (\mid - -)\ d\}$$

$$(a \prec b) \times (d \prec c) \prec \{a\ (\mid - \mid)\ c \prec b\ (\mid - \mid)\ d\}$$
$$(a \prec b) \times (d \prec c) \prec \{a\ (-\mid -)\ c \prec b\ (-\mid -)\ d\}$$

$$(b \prec a) \times (d \prec c) \prec \{a\ (- - \mid)\ c \prec b\ (- - \mid)\ d\}$$
$$(b \prec a) \times (d \prec c) \prec \{a\ (\mid\mid -)\ c \prec b\ (\mid\mid -)\ d\}.$$

As it is inconvenient to consider so many as eight distinct operations, we may reject one-half of these so as to retain one under each of the four rules. We may reject all those whose symbols contain an odd number of hyphens (as being negative). We then retain four, to which we may assign the following names and symbols:

$a\ (\mid\mid\mid)\ e = ae$ *Relative or external multiplication.*
$a\ (\mid - -)\ e = {}^{a}e$ *Regressive involution.*
$a\ (-\mid -)\ e = a^{e}$ *Progressive involution.*
$a\ (- - \mid)\ e = a \circ e$ *Transaddition.**

We have then the following table of equivalents, negatives, and converses: †

x	$\bar{x}$	$\breve{x}$	$\breve{\bar{x}}$
$ae = \bar{a} \circ \bar{e}$	$\bar{a}^{e} = {}^{a}\bar{e}$	$\breve{e}\breve{a} = \breve{\bar{e}} \circ \breve{\bar{a}}$	$\breve{\bar{e}}^{\breve{a}} = {}^{\breve{e}}\breve{\bar{a}}$
$a^{e} = {}^{\bar{a}}\bar{e}$	$\bar{a}e = a \circ \bar{e}$	${}^{\breve{e}}\breve{a} = \breve{\bar{e}}^{\breve{\bar{a}}}$	$\breve{e}\breve{\bar{a}} = \breve{\bar{e}} \circ \breve{a}$
${}^{a}e = \bar{a}^{\bar{e}}$	$a\bar{e} = \bar{a} \circ e$	$\breve{e}^{\breve{a}} = {}^{\breve{\bar{e}}}\breve{\bar{a}}$	$\breve{\bar{e}}\breve{a} = \breve{e} \circ \breve{\bar{a}}$
$a \circ e = \bar{a}\bar{e}$	$a^{\bar{e}} = {}^{\bar{a}}e$	$\breve{e} \circ \breve{a} = \breve{\bar{e}}\breve{\bar{a}}$	$\breve{e}^{\breve{\bar{a}}} = {}^{\breve{\bar{e}}}\breve{a}$

* The first three of these were studied by De Morgan (*On the Syllogism*, No. IV.); the last is new. The above names for the first three (except the adjective *internal* suggested by Grassmann's operation) are given in my *Logic of Relatives*.

† A similar table is given by De Morgan. Of course, it lacks the symmetry of this, because he had not the fourth operation.

If l denote 'lover' and s 'servant,' then

ls denotes whatever is lover of a servant of –,

l^s whatever is lover of every servant of –,

ls whatever is in every way (in which it loves at all) lover of a servant,

$l \circ s$ whatever is not a non-lover only of a servant of –

or whatever is not a lover of everything but servants of –

or whatever is some way a non-lover of some thing besides servants of –.

§ 6. *Methods in the Algebra of Relatives.*

The universal method in this algebra is the method of limits. For certain letters are to be substituted an infinite sum of individuals or product of simples; whereupon certain transformations become possible which could not otherwise be effected.

The following theorems are indispensable for the application of this method:

1st. $$l^{\mathrm{A:B}} = l(\mathrm{A:B}) + k\bar{\mathrm{B}}.$$

Since $\bar{\mathrm{B}}$ is equivalent to the relative term which comprises all individual relatives whose relates are not B, so $k\bar{\mathrm{B}}$ may be conveniently used, as it is here, to express the aggregate of all individual relatives whose correlate is $\bar{\mathrm{B}}$. To prove this proposition, we observe that

$$l^{\mathrm{A:B}} = \overline{\bar{l}(\mathrm{A:B})}.$$

Now $\bar{l}(\mathrm{A:B})$ contains only individual relatives whose correlate is B, and of these it contains those which are not included in $l(\mathrm{A:B})$. Hence the negative of $\bar{l}(\mathrm{A:B})$ contains all individual relatives whose correlates are not B, together with all contained in $l(\mathrm{A:B})$. Q. E. D.

2d. $${}^{\mathrm{A:B}}l = (\mathrm{A:B})l + \bar{\mathrm{A}}.$$

Here $\bar{\mathrm{A}}$ is used to denote the aggregate of all individual relatives whose relates are not A. This proposition is proved like the last.

3d. $$\overline{\mathrm{A:B}}^{l} = (\mathrm{A:B})\,\bar{l} + \bar{\mathrm{A}}.$$

This is evident from the second proposition, because

$$\overline{\mathrm{A:B}}^{l} = {}^{(\mathrm{A:B})}\bar{l}.$$

4th. $${}^{l}\overline{\mathrm{A:B}} = \bar{l}(\mathrm{A:B}) + k\bar{\mathrm{B}}.$$

Another method of working with the algebra is by means of negations. This becomes quite indispensable when the operations are defined by negations, as in this paper.

To illustrate the use of these methods, let us investigate the relations of ${}^lb$ and l^b to lb when l and b are totally unlimited relatives.

Write $$l = \Sigma_i (\mathrm{L}_i : \mathrm{M}_i) \qquad b = \Sigma_j (\mathrm{B}_j : \mathrm{C}_j).$$

Then, by the rules of the last section,

$${}^lb \prec {}^{\mathrm{L}:\mathrm{M}}b \qquad l^b \prec l^{\mathrm{B}:\mathrm{C}};$$

whence, by the second and third propositions above,

$${}^lb \prec (\mathrm{L}_i : \mathrm{M}_i)\, b + \bar{\mathrm{L}}_i \qquad l^b \prec l\,(\mathrm{B}_j : \mathrm{C}_j) + k\bar{\mathrm{B}}_j.$$

But by the first rule of the last section

$$(\mathrm{L}_i : \mathrm{M}_i)\, b \prec lb \qquad l\,(\mathrm{B}_j : \mathrm{C}_j) \prec lb\,;$$

hence,

$${}^lb \prec lb + \bar{\mathrm{L}}_i \qquad l^b \prec lb + k\bar{\mathrm{B}}_j.$$

There will be propositions like these for all the different values of i and j. Multiplying together all those of the several sets, we have

$${}^lb \prec lb + \Pi_i\bar{\mathrm{L}}_i \qquad l^b \prec lb + \Pi_j k\bar{\mathrm{B}}_j.$$

But

$$\Pi_i\bar{\mathrm{L}}_i = \overline{\Sigma_i\mathrm{L}_i} \qquad \Pi_j k\bar{\mathrm{B}}_j = \overline{\Sigma_j k\mathrm{B}_j},$$

and since the relatives are unlimited,

$$\Sigma_i\mathrm{L}_i = \infty \qquad \Sigma_j k\mathrm{B}_j = \infty$$
$$\overline{\Sigma_i\mathrm{L}_i} = 0 \qquad \overline{\Sigma_j k\mathrm{B}_j} = 0\,;$$

hence

$${}^lb \prec lb \qquad l^b \prec lb.$$

In the same way it is easy to show that, if the negatives of l and b are totally unlimited,

$$l^b \prec l \circ b \qquad {}^lb \prec l \circ b.$$

§ 7. *The General Formulæ for Relatives.*

The principal formulæ of this algebra may be divided into *distribution formulæ* and *association formulæ.* The distribution formulæ are those which give the equivalent of a relative compounded with a sum or product of two relatives in such terms as to separate the latter two relatives. The association formulæ are those which give the equivalent of a relative A compounded with a compound of B and C in terms of a compound of A and B compounded with C.

I. DISTRIBUTION FORMULÆ.

1. AFFIRMATIVE.

i. *Simple Formulæ.*

$$(a+b)\,c = ac + bc \qquad a\,(b+c) = ab + ac$$

$$(a \times b)^c = a^c \times b^c \qquad a^{b+c} = a^b \times a^c$$

$${}^{a+b}c = {}^{a}c \times {}^{b}c \qquad {}^{a}(b \times c) = {}^{a}b \times {}^{a}c$$

$$(a \times b) \circ c = (a \circ c) + (b \circ c) \qquad a \circ (b \times c) = (a \circ b) + (a \circ c)$$

ii. *Developments.*

$$(a \times b)\,c = \Pi_p \,\{\, a\,(c \times p) + b\,(c \times \bar{p}) \,\} \qquad a\,(b \times c) = \Pi_p \,\{\, (a \times p)\,b + (a \times \bar{p})\,c \}$$

$$(a+b)^c = \Sigma_p \,\{\, a^{c \times p} \times b^{c \times \bar{p}} \,\} \qquad a^{b \times c} = \Sigma_p \,\{\, (a+p)^b \times (a+\bar{p})^c \}$$

$${}^{(a \times b)}c = \Sigma_p \,\{\, {}^{a}(c+p) \times {}^{b}(c+\bar{p}) \,\} \qquad {}^{a}(b+c) = \Sigma_p \,\{\, {}^{a \times p}b \times {}^{a \times \bar{p}}c \}$$

$$(a+b) \circ c = \Pi_p \{ a \circ (c+p) + b \circ (c+\bar{p}) \} \qquad a \circ (b+c) = \Pi_p \{ (a+p) \circ b + (a+\bar{p}) \circ c \}$$

2. NEGATIVE.

i. *Simple Formulæ.*

$$\overline{(a+b)\,c} = \overline{ac} \times \overline{bc} \qquad \overline{a\,(b+c)} = \overline{ab} + \overline{ac}$$

$$\overline{(a \times b)^c} = \overline{a^c} + \overline{b^c} \qquad \overline{a^{b+c}} = \overline{a^b} + \overline{a^c}$$

$$\overline{{}^{a+b}c} = \overline{{}^{a}c} + \overline{{}^{b}c} \qquad \overline{{}^{a}(b \times c)} = \overline{{}^{a}b} + \overline{{}^{a}c}$$

$$\overline{(a \times b) \circ c} = \overline{a \circ c} \times \overline{b \circ c} \qquad \overline{a \circ (b \times c)} = \overline{a \circ b} \times \overline{a \circ c}$$

ii. *Developments.*

$$\overline{(a \times b)\,c} = \Sigma_p \,\{ \overline{a\,(c \times p)} \times \overline{b\,(c \times \bar{p})} \} \qquad \overline{a\,(b \times c)} = \Sigma_p \,\{\, \overline{(a \times p)\,b} \times \overline{(a \times \bar{p})\,c} \}$$

$$\overline{(a+b)^c} = \Pi_p \,\{\, \overline{a^{c \times p}} + \overline{b^{c \times p}} \,\} \qquad \overline{a^{b \times c}} = \Pi_p \,\{\, \overline{(a+p)^b} + \overline{(a+\bar{p})^c} \}$$

$$\overline{{}^{(a \times b)}c} = \Pi_p \,\{\, \overline{{}^{a}(c+p)} + \overline{{}^{b}(c+\bar{p})} \} \qquad \overline{{}^{a}(b+c)} = \Pi_p \,\{\, \overline{{}^{a \times p}b} + \overline{{}^{a \times \bar{p}}c} \}$$

$$\overline{(a+b) \circ c} = \Sigma_p \{ \overline{a \circ (c+p)} \times \overline{b \circ (c+\bar{p})} \} \qquad \overline{a \circ (b+c)} = \Sigma_p \{ \overline{(a+p) \circ b} \times \overline{(a+\bar{p}) \circ c} \}$$

II. ASSOCIATION FORMULÆ.

1. AFFIRMATIVE.

i. *Simple Formulæ.*

$$\overline{{}^{a}(\overline{bc})} = a\,(bc) = (ab)\,c = \overline{(\overline{ab})^{c}} \qquad \overline{a(\overline{b^{c}})} = {}^{a}(b^{c}) = ({}^{a}b)^{c} = \overline{(\overline{{}^{a}b})\,c}$$

$$\overline{a\circ(\overline{b\circ c})} = a^{(b\circ c)} = {}^{(a\circ b)}c = \overline{(\overline{a\circ b})\circ c} \qquad \overline{a^{(\overline{{}^{b}c})}} = a\circ({}^{b}c) = (a^{b})\circ c = \overline{{}^{(\overline{a^{b}})}c}$$

$$\overline{a\circ(\overline{bc})} = a^{(bc)} = (a^{b})^{c} = \overline{(\overline{a^{b}})\,c} \qquad \overline{a\,(\overline{{}^{b}c})} = {}^{a}({}^{b}c) = {}^{(ab)}c = \overline{(\overline{ab})\circ c}$$

$$\overline{{}^{a}(\overline{b\circ c})} = a\,(b\circ c) = ({}^{a}b)\circ c = \overline{{}^{(\overline{ab})}c} \qquad \overline{a^{(\overline{b^{c}})}} = a\circ(b^{c}) = (a\circ b)\,c = \overline{(\overline{a\circ b})^{c}}$$

ii. *Developments.*

(A and E are individual aggregants, and α and ϵ simple components of a and e. The summations and products are relative to all such aggregants and components. The formulæ are of four classes; and for any relative c either all formulæ of Class 1 or all of Class 2, and also either all of Class 3 or all of Class 4 hold good.

CLASS 1.

$$\overline{a(\overline{bc})} = {}^{a}(bc) = \Pi\{({}^{\mathrm{A}}b)c\} = \Pi\{\overline{(\overline{{}^{\mathrm{A}}b})^{c}}\}$$
$$\overline{a^{(\overline{bc})}} = a\circ(bc) = \Sigma\{(\alpha\circ b)^{c}\} = \Sigma\{\overline{(\overline{\alpha\circ b})c}\}$$
$$\overline{a\circ(\overline{b^{c}})} = a^{(b^{c})} = \Pi\{(\alpha^{b})c\} = \Pi\{\overline{(\overline{\alpha^{b}})^{c}}\}$$
$$\overline{{}^{a}(\overline{b^{c}})} = a(b^{c}) = \Sigma\{(\mathrm{A}b)^{c}\} = \Sigma\{\overline{(\overline{\mathrm{A}b})c}\}$$

CLASS 2.

$$\overline{(\overline{c\circ d})e} = (c\circ d)^{e} = \Pi\{c\circ(d\mathrm{E})\} = \Pi\{\overline{c^{(\overline{d\mathrm{E}})}}\}$$
$$\overline{{}^{(\overline{c\circ d})}e} = (c\circ d)\circ e = \Sigma\{c^{({}^{d}e)}\} = \Sigma\{\overline{c\circ(\overline{{}^{d}\epsilon})}\}$$
$$\overline{(\overline{c^{d}})\circ e} = {}^{(c^{d})}e = \Pi\{c\circ(d\circ\epsilon)\} = \Pi\{\overline{c^{(\overline{d\circ\epsilon})}}\}$$
$$\overline{(\overline{c^{d}})^{e}} = (c^{d})e = \Sigma\{c^{(d^{\mathrm{E}})}\} = \Sigma\{\overline{c\circ(\overline{d^{\mathrm{E}}})}\}$$

CLASS 3.

$$\overline{a^{(\overline{b\circ c})}} = a\circ(b\circ c) = \Sigma\{({}^{(\alpha^{h})}c\} = \Sigma\{\overline{(\overline{\alpha^{b}})c}\}$$
$$\overline{a(\overline{b\circ c})} = {}^{a}(b\circ c) = \Pi\{(\mathrm{A}b)\circ c\} = \Pi\{\overline{{}^{(\overline{\mathrm{A}b})}c}\}$$
$$\overline{{}^{a}(\overline{{}^{b}c})} = a({}^{b}c) = \Sigma\{{}^{(\mathrm{A}b)}c\} = \Sigma\{\overline{(\overline{{}^{\mathrm{A}}b})\circ c}\}$$
$$\overline{a\circ(\overline{{}^{b}c})} = a^{{}^{b}c)} = \Pi\{(\alpha\circ b)\circ c\} = \Pi\{\overline{{}^{(\overline{\alpha\circ b})}c}\}$$

CLASS 4.

$$\overline{{}^{(cd)}e} = (cd)\circ e = \Sigma\{{}^{c}(d\circ\epsilon)\} = \Sigma\{\overline{c(\overline{d\circ\epsilon})}\}$$
$$\overline{(\overline{cd})e} = (cd)^{e} = \Pi\{c(d^{\mathrm{E}})\} = \Pi\{\overline{{}^{c}(\overline{d^{\mathrm{E}}})}\}$$
$$\overline{(\overline{{}^{c}d})^{e}} = ({}^{c}d)e = \Sigma\{{}^{c}(d\mathrm{E})\} = \Sigma\{\overline{c(\overline{d\mathrm{E}})}\}$$
$$\overline{(\overline{{}^{c}d})\circ e} = {}^{({}^{c}d)}e = \Pi\{c({}^{d}\epsilon)\} = \Pi\{\overline{{}^{c}(\overline{{}^{d}\epsilon})}\}$$

The negative formulæ are derived from the affirmative by simply drawing or erasing lines over the whole of each member of every equation.

In order to see the general rules which these formulæ follow, we must imagine the operations symbolized by three marks, as in the commencement of this chapter. We may term the operation uniting the two letters within the parenthesis the *interior* operation, and that which unites the whole parenthesis to the third letter the *exterior* operation. By *junction-marks* will be meant, in case the parenthesis $\left\{\begin{matrix}\text{follows}\\ \text{precedes}\end{matrix}\right\}$ the third letter, the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ mark of the symbol of the interior operation and the $\left\{\begin{matrix}\text{second}\\ \text{first}\end{matrix}\right\}$ mark of the symbol of the exterior operation. Using these terms, we may say that the exterior junction-mark and the third mark of the interior operation may always be changed together. When they are the same there is a simple association formula. This formula consists in the possibility of simultaneously interchanging the junction-marks, the third marks, and the exteriority or interiority of the two operations. When the exterior junction-mark and the third mark of the interior operation are unlike, there is a developmental association formula. The general term of this formula is obtained by making the same interchanges as in the simple formulæ, and then changing a to A when after these interchanges ab or ${}^ab$ occurs in parenthesis, changing a to $\boldsymbol{\alpha}$ when a^b or $a \circ b$ occurs in parenthesis, changing e to E when de or d^e occurs in parenthesis, and changing e to ϵ when ${}^de$ or $d \circ e$ occurs in parenthesis. When the third mark in the symbol of the exterior operation is affirmative the development is a summation; when this mark is negative there is a continued product.

In the first column of formulæ, the second mark in the sign of the interior operation is a line in Class 1 and a hyphen in Class 3. In the second column, the first mark in the sign of the interior operation is a hyphen in Class 2 and a line in Class 4.

(*To be Continued.*)

NOTE TO PAGE 47.

The relative 0 ought to be considered as at once a concurrent and an alio-relative, and the relative ∞ as at once the negative of a concurrent and the negative of an alio-relative. The statements in the text require to be modified to this extent.